arXiv:1112.4080v1 [nlin.SI] 17 Dec 2011

Stability of periodic traveling waves for the quadratic and cubic nonlinear Schrödinger equations

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December 20, 2011

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Abstract

We study the existence and stability of periodic traveling-wave solutions for the quadratic and cubic nonlinear Schrödinger equations in one space dimension.

1 Introduction

In this work we consider the nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + |u|^p u = 0. \tag{1.1}$$

This equation appears in various problems, modeling many phenomena such as the behavior of a non-ideal Bose gas with a weak particle interaction, the spreading of the heat impulse in solids, the Langmuir waves in a plasma, etc. [17, 18].

Our principal aim is to study the the orbital stability of the family of periodic traveling-wave solutions

$$u = \varphi(x,t) = e^{i(vx+(\omega-v^2)t)} r(x-2vt). \tag{1.2}$$

where $r(y)$ is a real-valued T-periodic function and $v, \omega \in \mathbb{R}$ are parameters, for quadratic (p=1) and cubic (p=2) nonlinear Schrödinger equation.The problem of the stability of solitary waves for nonlinear dispersive equations goes back to the works of Benjamin [4] and Bona [5] (see also [1, 15, 16]). A general approach for investigating the stability of solitary waves for nonlinear equations having a group of symmetries was proposed in [8]. The existence and stability of solitary wave solutions for equation (1.1) has been studied in [19]. Recently in [3], the authors developed a

complete theory on the stability of cnoidal waves for the KdV equation. Other new explicit formulae for the periodic traveling waves based on the Jacobi elliptic functions, together with their stability, have been obtained in [2, 9, 10] for the nonlinear Schrödinger equation, modified KdV equation, complex modified KdV equation, and generalized BBM equation. In [11], the stability of periodic traveling wave solutions of BBM equation which wave profile stays close to the constant state $u = (c-1)^{1/p}$ is considered.

In this paper, we prove stability of the periodic traveling waves (1.2) not oscillating around zero ($r \neq 0$) for the quadratic and the cubic NLS. Our main results are formulated as Theorem 2.1 and Theorem 3.1 in Sections 2 and 3 below. We base our analysis on some appropriate invariant laws. Our approach is to verify that φ is a minimizer of a properly chosen functional M which is conservative with respect to time over the solutions of (1.1). We consider the L^2-space of T-periodic functions in $x \in \mathbb{R}$, with a norm $||.||$ and a scalar product $\langle .,. \rangle$. To establish that the orbit

$$\mathcal{O} = \{e^{i\eta}\varphi(\cdot - \xi, t) : \ (\xi, \eta) \in [0, T] \times [0, 2\pi]\}$$

is stable, we take

$$u(x,t) = e^{i\eta}\varphi(x-\xi,t) + h(x,t) = e^{i\zeta}[r(x-\xi-2vt) + h_1 + ih_2]$$

and express the leading term of $M(u) - M(\varphi)$ as $\langle L_1 h_1, h_1\rangle + \langle L_2 h_2, h_2\rangle$ where L_i are second-order selfadjoint differential operators in $L^2[0,T]$ with potentials depending on r and satisfying $L_1 r' = L_2 r = 0$. The proof of orbital stability requires that zero is the second eigenvalue of L_1 and the first one of L_2.

Recall that the quadratic and cubic nonlinear Schrödinger equations are globally well-posed in $H^s(\mathbb{T})$, for $s \geq 0$ [6].

The paper is organized as follows. We consider the quadratic and the cubic cases in Sections 2 and 3 respectively. In Appendices 1 and 2, some facts from the theory of complete Abelian integrals (e.g. Picard-Fuchs equations, polynomial moduli) are used in order to derive several inequalities we needed during the proof our main results.

2 Existence and stability of periodic traveling waves for the quadratic Schrödinger equation

Consider the equation

$$iu_t + u_{xx} + |u|u = 0, \tag{2.1}$$

where u is a complex-valued function.

We are looking for a solution of equation (2.1) in the form (1.2) where r is real-valued. For r one obtains the equation

$$r'' - \omega r + r|r| = 0. \tag{2.2}$$

Therefore,

$$r'^2 - \omega r^2 + \frac{2}{3}r^2|r| = c \tag{2.3}$$

and r is periodic provided that the level set $H(x,y)=c$ of the Hamiltonian system $dH=0$, with

$$H(x,y)=y^2-\omega x^2+\frac{2}{3}x^2|x|,$$

contains a periodic trajectory (an oval). The level set $H(x,y)=c$ contains two periodic trajectories if $\omega>0$, $c\in(-\frac{1}{3}\omega^3,0)$ and a unique periodic trajectory if $\omega\in\mathbb{R}$, $c>0$. Under these conditions, equation (2.3) becomes $H(r,r')=c$ and its solution r is periodic of period $T=T(\omega,c)$.

Below, we consider the case $c<0$. Then either $r<0$ (the left case) or $r>0$ (the right case). To express r through elliptic functions, we denote by $r_0>r_1>0$ the positive solutions of $\frac{2}{3}\rho^3-\omega\rho^2-c=0$. Then $r_1\le|r|\le r_0$ and one can rewrite (2.3) as

$$r'^2=\tfrac{2}{3}(|r|-r_1)(r_0-|r|)(|r|+r_0+r_1-\tfrac{3}{2}\omega). \tag{2.4}$$

Therefore $2r_0+r_1>r_0+2r_1>\frac{3}{2}\omega$. Introducing a new variable $s\in(0,1)$ via $|r|=r_1+(r_0-r_1)s^2$, we transform (2.4) into

$$s'^2=\alpha^2(1-s^2)(k'^2+k^2s^2)$$

where α, k, k' are positive constants ($k^2+k'^2=1$) given by

$$\alpha^2=\frac{4r_0+2r_1-3\omega}{12},\quad k^2=\frac{2r_0-2r_1}{4r_0+2r_1-3\omega},\quad k'^2=\frac{2r_0+4r_1-3\omega}{4r_0+2r_1-3\omega}.$$

Therefore

$$|r(x)|=r_1+(r_0-r_1)cn^2(\alpha x;k). \tag{2.5}$$

$$T=\frac{2K(k)}{\alpha}=\frac{4\sqrt[4]{1-k^2+k^4}K(k)}{\sqrt{\omega}},\ k\in(0,1),\ T\in I=\left(\frac{2\pi}{\sqrt{\omega}},\infty\right). \tag{2.6}$$

As usual, here and below, $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and the second kind in a Legendre form. Let us recall for later use the system they satisfy:

$$kK'=\frac{E}{1-k^2}-K,\ \ kE'=E-K.$$

Lemma 2.1. *For any $\omega>0$ and $T\in I$, there is a constant $c=c(\omega)$ such that the periodic traveling-wave solution* (2.5) *determined by $H(r,r')=c(\omega)$ has a period T. The function $c(\omega)$ is differentiable.*

Proof. The statement follows from the implicit function theorem. It is easily seen that the period T is a strictly increasing function of k:

$$\begin{aligned}\tfrac{d}{dk}(\sqrt[4]{1-k^2+k^4}K(k)) &= \tfrac{k(2k^2-1)K(k)+2(1-k^2+k^4)K'(k)}{2(1-k^2+k^4)^{3/4}}\\ &= \tfrac{2(1-k^2+k^4)E(k)+(1-k^2)(k^2-2)K(k)}{2k(1-k^2)(1-k^2+k^4)^{3/4}}>0.\end{aligned}$$

Given ω and c in their range, consider the functions $r_0(\omega,c), r_1(\omega,c)$, $k(\omega,c)$ and $T(\omega,c)$ given by the formulas we derived above. We obtain

$$\frac{\partial T}{\partial c}=\frac{dT}{dk}\frac{dk}{dc}=\frac{1}{2k}\frac{dT}{dk}\frac{d(k^2)}{dc}.$$

Further, using that $k^2=3\frac{2r_0-\omega}{4r_0+2r_1-3\omega}-1$ and $\frac{2}{3}r_0^3-\omega r_0^2=\frac{2}{3}r_1^3-\omega r_1^2$, we have

$$\begin{aligned}\frac{d(k^2)}{dc} &= 3\frac{(4r_1-2\omega)\frac{\partial r_0}{\partial c}-(4r_0-2\omega)\frac{\partial r_1}{\partial c}}{(4r_0+2r_1-3\omega)^2}\\ &= \frac{3\omega^2(r_1-r_0)}{4(r_0^2-\omega^2)(r_1^2-\omega^2)(4r_0+2r_1-3\omega)^2}.\end{aligned}$$

We see that $\partial T(\omega,c)/\partial c\neq 0$, therefore the implicit function theorem yields the result. □

Equation (2.1) has the following conservation laws

$$Q(u)=i\int_0^T \overline{u}_x u dx,\quad P(u)=\int_0^T |u|^2dx,\quad E(u)=\int_0^T (|u_x|^2-\frac{2|u|^3}{3})dx.$$

Let us consider the functional

$$M(u)=E(u)+(\omega+v^2)P(u)-2vQ(u).$$

Next we introduce the pseudometric

$$d(u,\varphi)=\inf_{(\eta,\xi)\in[0,2\pi]\times[0,T]}||u(x,t)-e^{i\eta}\varphi(x-\xi,t)||_1. \tag{2.7}$$

For a fixed $q>0$, we denote

$$d_q^2(u,\varphi)=\inf_{(\eta,\xi)\in[0,2\pi]\times[0,T]}(||u_x(x,t)-e^{i\eta}\varphi_x(x-\xi,t)||^2+q||u(x,t)-e^{i\eta}\varphi(x-\xi,t)||^2). \tag{2.8}$$

Clearly, the infimum in (2.8) is attained at some point (η,ξ) in the square $[0,T]\times[0,T]$. Moreover, for $q\in[q_1,q_2]\subset(0,\infty)$, (2.8) is a pseudometric equivalent to (2.7).

Lemma 2.2. *The metric $d_q(u,\varphi)$ is a continuous function of $t\in[0,\infty)$.*

Proof. The proof of the lemma is similar to the proof of Lemmas 1, 2 in [5] □.

Now, we can formulate our main result in this section.

Theorem 2.1. *Let φ be given by (1.2), with $r\neq 0$. For each $\varepsilon>0$ there exists $\delta>0$ such that if $u(x,t)$ is a solution of (2.1) and $d(u,\varphi)_{|t=0}<\delta$, then $d(u,\varphi)<\varepsilon$ $\forall t\in[0,\infty)$.*

The crucial step in the proof will be to verify the following statement.

Proposition 2.1. *There exist positive constants m,q,δ_0 such that if u is a periodic solution of (2.1), $u(x,t)=u(x+T,t)$, $P(u)=P(\varphi)$ and $d_q(u,\varphi)<\delta_0$, then*

$$M(u)-M(\varphi)\geq md_q^2(u,\varphi). \tag{2.9}$$

Proof. In order to estimate $\Delta M = M(u) - M(\varphi)$, we set

$$u(x,t) = e^{i\eta}\varphi(x-\xi,t) + h(x,t),$$

$$\zeta = v(x-\xi) + (\omega - v^2)t + \eta, \quad F(s) = -\frac{2}{3}|e^{i\eta}\varphi + hs|^3.$$

We have

$$\frac{2}{3}(-|u|^3 + |\varphi|^3) = F(1) - F(0) = F'(0) + \frac{F''(0)}{2} + \frac{F''(s) - F''(0)}{2}$$

where

$$0 \le s \le 1, \ F'(0) = -2|\varphi| Re(e^{i\eta}\varphi\overline{h}), \ F''(0) = -|\varphi| \left[Re(h^2 e^{-2i\zeta}) + 3|h|^2\right].$$

Integrating by parts in the terms containing h_x and $\overline{h}_x$, we obtain

$$\begin{aligned}
\Delta M &= M(u) - M(\varphi) \\
&= 2Re \int_0^T e^{i\eta}[-\varphi_{xx} + (\omega + v^2 - |\varphi|)\varphi + 2iv\varphi_x]\overline{h}dx \\
&+ \int_0^T [|h_x|^2 + (\omega + v^2 - \tfrac{3}{2}|\varphi|)|h|^2 - \tfrac{|\varphi|}{2}Re(e^{-2i\zeta}h^2) - 2ivh\overline{h}_x]dx \\
&- \int_0^T \tfrac{F''(s) - F''(0)}{2}dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Using that $r(x)$ satisfies the equation (2.2) we obtain that $I_1 = 0$.

Let $h(x,t) = (h_1 + ih_2)e^{i\zeta}$, where h_1 and h_2 are real periodic functions with period T. Then we have

$$\begin{aligned}
&|h|^2 = h_1^2 + h_2^2 \\
&|h_x|^2 = h_{1x}^2 + h_{2x}^2 + 2v(h_1h_{2x} - h_{1x}h_2) + v^2(h_1^2 + h_2^2) \\
&Re(e^{-2i\zeta}h^2) = h_1^2 - h_2^2 \\
&\int_0^T h\overline{h}_x dx = i\int_0^T (h_{1x}h_2 - h_1h_{2x} - vh_1^2 - vh_2^2)dx.
\end{aligned} \tag{2.10}$$

Finally, for I_2 we obtain

$$I_2 = \int_0^T [h_{1x}^2 + (\omega - 2|r|)h_1^2]dx + \int_0^T [h_{2x}^2 + (\omega - |r|)h_2^2]dx = M_1 + M_2$$

Consider in $[0,T]=[0,2K(k)/\alpha]$ the formal differential operators

$$L_1=-\frac{d^2}{dx^2}+(\omega-2|r|),\quad L_2=-\frac{d^2}{dx^2}+(\omega-|r|),$$

supplied with periodic boundary conditions. By the above formulas, $r_0-r_1=6\alpha^2k^2$, $2r_0-\omega=4\alpha^2(1+k^2)$. Taking $y=\alpha x$ as an independent variable in L_1, one obtains $L_1=\alpha^2\Lambda_1$ with an operator Λ_1 in $[0,2K(k)]$ given by

$$\begin{aligned}\Lambda_1 &= -\frac{d^2}{dy^2}+\alpha^{-2}[\omega-2(r_1+(r_0-r_1)cn^2(y;k))]\\ &= -\frac{d^2}{dy^2}+\frac{\omega-2r_0}{\alpha^2}+\frac{2(r_0-r_1)}{\alpha^2}sn^2(y;k)\\ &= -\frac{d^2}{dy^2}-4(1+k^2)+12k^2sn^2(y;k).\end{aligned}$$

The spectral properties of the operator Λ_1 in $[0,2K(k)]$ are well known. The first three(simple) eigenvalues and corresponding eigenfunctions of Λ_1 are

$$\begin{aligned}&\mu_0=k^2-2-2\sqrt{1-k^2+4k^4}<0,\\ &\psi_0(y)=dn(y;k)[1-(1+2k^2-\sqrt{1-k^2+4k^4})sn^2(y;k)]>0\end{aligned}$$

$$\begin{aligned}&\mu_1=0\\ &\psi_1(y)=dn(y;k)sn(y;k)cn(y;k)=\tfrac{1}{2}\tfrac{d}{dy}cn^2(y;k)\end{aligned}$$

$$\begin{aligned}&\mu_2=k^2-2+2\sqrt{1-k^2+4k^4}>0\\ &\psi_2(y)=dn(y;k)[1-(1+2k^2+\sqrt{1-k^2+4k^4})sn^2(y;k)].\end{aligned}$$

Since the eigenvalues of L_1 and Λ_1 are related by $\lambda_n=\alpha^2\mu_n$, it follows that the first three eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0,2K(k)]$ are simple and $\lambda_0<0,\lambda_1=0,\lambda_2>0$. The corresponding eigenfunctions are $\psi_0(\alpha x),\psi_1(\alpha x)=const.r'$ and $\psi_2(\alpha x)$.

In a similar way, with $L_2=\alpha^2\Lambda_2$, one obtains in $[0,2K(k)]$

$$\Lambda_2=-\frac{d^2}{dy^2}-2(1+k^2)+6k^2sn^2(y;k)+\omega/2\alpha^2.$$

To express ω through α and k, one should take into account the fact that in the cubic equation we used to determine r_0 and r_1, the coefficient at ρ is zero. Therefore,

$$r_0r_1+(r_0+r_1)(\tfrac{3}{2}\omega-r_0-r_1)=0.$$

As $r_0=2\alpha^2+2\alpha^2k^2+\frac{1}{2}\omega$, $r_1=2\alpha^2-4\alpha^2k^2+\frac{1}{2}\omega$, after replacing these values in the above equation one obtains $\omega^2=16\alpha^4(1-k^2+k^4)$. Since $\omega>0$, we finally obtain

$$\Lambda_2=-\frac{d^2}{dy^2}+2(-1-k^2+\sqrt{1-k^2+k^4})+6k^2sn^2(y;k).$$

On the other hand, (2.5) yields

$$|r| = 2\alpha^2[1 + k^2 + \sqrt{1 - k^2 + k^4} - 3k^2 sn^2(y;k)].$$

The first three eigenvalues and corresponding eigenfunctions of Λ_2 are as follows:

$$\begin{aligned}
&\epsilon_0 = 0, && \upsilon_0 = r,\\
&\epsilon_1 = 2 - k^2 + 2\sqrt{1 - k^2 + k^4}, && \upsilon_1 = dn'(y;k)\\
&\epsilon_2 = 4\sqrt{1 - k^2 + k^4}, && \upsilon_2 = 1 + k^2 - \sqrt{1 - k^2 + k^4} - 3k^2 sn^2(y;k).
\end{aligned}$$

Estimates for M_2.

From the above explanations we know that when considered in $[0, T]$, the operator L_2 has an eigenfunction r corresponding to zero eigenvalue and the rest of the spectrum is contained in $(\alpha^2\lambda_1, \infty)$.

In the formulas which follow, we take $r = r(\bar{x})$ with an argument $\bar{x} = x - \xi - 2vt$. The values of ξ and η are chosen so that the infimum in (2.8) is attained at that point. Therefore the derivative of $d_q^2(u, \varphi)$ with respect to η is equal to zero. Together with (2.2), this yields

$$\begin{aligned}
0 &= i\int_0^T h_x e^{-i\eta}\overline{\varphi_x} - \varphi_x e^{i\eta}\overline{h_x} + q(he^{-i\eta}\overline{\varphi} - \varphi e^{i\eta}\overline{h})dx\\
&= 2Im\int_0^T (-\varphi_{xx} + q\varphi)e^{i\eta}\overline{h}dx \qquad (2.11)\\
&= \int_0^T [(v^2 - \omega + q + |r|)\, rh_2 + 2vr'h_1]dx.
\end{aligned}$$

We set $h_2 = \beta r(\bar{x}) + \theta$, $\int_0^T \theta r dx = 0$. Substituting in (2.11), we obtain

$$\beta||r||^2\left[v^2 - \omega + q + \frac{||r^{3/2}||^2}{||r||^2}\right] + \int_0^T [\theta r|r| + 2vr'h_1]dx = 0.$$

Using that $\frac{||r^{3/2}||^2}{||r||^2} > \omega$ (see estimate A1 of Appendix 1), we obtain the estimate

$$\begin{aligned}
|\beta|\,||r|| &\le \frac{\left|\int_0^T(\theta r|r| + 2vr'h_1)dx\right|}{(v^2 + q)||r||}\\
&\le \frac{||r^2||\cdot||\theta|| + 2|v|\,||r'||\cdot||h_1||}{(v^2 + q)||r||}\\
&\le m_0(||\theta|| + ||h_1||),
\end{aligned}$$

where $m_0 = 2m_1(v,\omega)/(q+\omega^2)$ and

$$m_1(v,\omega) = \max_{c\in[-\frac{1}{3}\omega^3,0]} \left(\frac{||r^2||}{||r||}, \frac{2|v|\,||r'||}{||r||}, \frac{2||\,|r|r'||}{||r'||}, \frac{|v|||\,|r|r-\omega r||}{||r'||} \right).$$

Clearly, the first and the third terms are uniformly bounded for ω fixed. The boundedness of the second and the fourth ones follows from the estimates in D1 of Appendix 1.(The third and fourth terms are included for later use.) We will use below that for v and ω fixed, $m_0 \to 0$ when $q \to \infty$. Further,

$$||h_2|| \le |\beta|\,||r|| + ||\theta|| \le m_0(||\theta|| + ||h_1||) + ||\theta|| = (m_0+1)||\theta|| + m_0||h_1||.$$

Hence, we obtain

$$||\theta||^2 \ge \frac{||h_2||^2}{2(m_0+1)^2} - \left(\frac{m_0}{m_0+1}\right)^2 ||h_1||^2. \tag{2.12}$$

Since $L_2 r = 0$ and $\langle\theta, r\rangle = 0$, then from the spectral properties of the operator L_2, it follows

$$M_2 = \langle L_2 h_2, h_2\rangle = \langle L_2\theta,\theta\rangle \ge \alpha^2\epsilon_1||\theta||^2.$$

From here and (2.12), one obtains

$$M_2 \ge \frac{\alpha^2\epsilon_1}{2(m_0+1)^2}||h_2||^2 - \frac{\alpha^2\epsilon_1 m_0^2}{(m_0+1)^2}||h_1||^2. \tag{2.13}$$

Estimates for M_1.

We set

$$h_1 = \gamma_1\psi_0(\bar{x}) + \gamma_2 r'(\bar{x}) + \theta_1, \quad r(\bar{x}) = \nu\psi_0(\bar{x}) + \psi, \tag{2.14}$$

where

$$\langle\theta_1,\psi_0\rangle = \langle\theta_1, r'\rangle = \langle\psi,\psi_0\rangle = \langle\psi_0, r'\rangle = \langle\psi, r'\rangle = 0 \tag{2.15}$$

and γ_1, γ_2 and ν are some constants. By (2.15), we have

$$M_1(h_1) = \langle L_1 h_1, h_1\rangle = \gamma_1^2\lambda_0\langle\psi_0,\psi_0\rangle + \langle L_1\theta_1,\theta_1\rangle.$$

Therefore, from spectral properties of the operator L_1 it follows

$$M_1(h_1) \ge \gamma_1^2\lambda_0||\psi_0||^2 + \lambda_2||\theta_1||^2. \tag{2.16}$$

The fundamental difficulty in the estimate of M_1 is the appearance of the negative term $\gamma_1^2\lambda_0||\psi_0||^2$. Below, we are going to estimate it. From the condition

$$P(u) = \int_0^T |h + e^{i\omega\eta}\varphi(x-\xi,t)|^2 dx = P(\varphi)$$

we obtain

$$||h||^2 = 2Re\int_0^T e^{i\omega\eta}\varphi(x-\xi,t)\overline{h}dx = -2\int_0^T r h_1 dx.$$

Then using (2.14), we have

$$-\frac{1}{2}||h||^2 = \nu\gamma_1||\psi_0||^2 + \int_0^T \psi\theta_1 dx$$

and therefore

$$\gamma_1^2||\psi_0||^2 = \frac{1}{\nu^2||\psi_0||^2}\left(\frac{1}{2}||h||^2 + \int_0^T \psi\theta_1 dx\right)^2. \tag{2.17}$$

From (2.17), we obtain

$$\gamma_1^2||\psi_0||^2 \leq \frac{1}{\nu^2||\psi_0||^2}\left(\frac{1+d}{4}||h||^4 + \frac{d+1}{d}||\psi||^2||\theta_1||^2\right), \tag{2.18}$$

where d is a positive constant which will be fixed later. Using (2.17) and (2.16), we derive the inequality

$$M_1 \geq \left(\lambda_2 + \lambda_0(1+\frac{1}{d})\frac{||\psi||^2}{\nu^2||\psi_0||^2}\right)||\theta_1||^2 + \frac{\lambda_0(1+d)}{4\nu^2||\psi_0||^2}||h||^4 \tag{2.19}$$

Below, we will denote by C_m, D_m positive constants. By using (2.18) and (2.19), we derive the inequality

$$\begin{aligned} M_1 &\geq \left(\lambda_2 + \lambda_0(1+\tfrac{1}{d})\tfrac{||\psi||^2}{\nu^2||\psi_0||^2}\right)||\theta_1||^2 + \tfrac{\lambda_0(1+d)}{4\nu^2||\psi_0||^2}||h||^4 \\ &\geq C_1\lambda_2||\theta_1||^2 - D_1||h||^4 \end{aligned} \tag{2.20}$$

(see the estimates in point C1 of the Appendix 1).

We denote $\vartheta = h_1 - \gamma_2 r'(\bar{x}) = \gamma_1\psi_0(\bar{x}) + \theta_1$. Then from (2.14), (2.20), we have

$$\begin{aligned} ||\vartheta||^2 = \gamma_1^2||\psi_0||^2 + ||\theta_1||^2 &\leq \left(1 + \tfrac{(d+1)||\psi||^2}{d\nu^2||\psi_0||^2}\right)||\theta_1||^2 + \tfrac{1+d}{4\nu^2||\psi_0||^2}||h||^4 \\ &\leq C_2||\theta_1||^2 + D_2||h||^4. \end{aligned}$$

Then

$$||\theta_1||^2 \geq \frac{||\vartheta||^2}{C_2} - \frac{D_2||h||^4}{C_2|a|^{\frac{1}{2}}}$$

and hence, by (2.20),

$$M_1 \geq \frac{C_1\lambda_2}{C_2}||\vartheta||^2 - \frac{C_1\lambda_2 D_2 + C_2 D_1}{C_2}||h||^4. \tag{2.21}$$

After differentiating (2.8) with respect to ξ and using (2.2), we obtain

$$\begin{aligned} 0 &= 2Re\int_0^T e^{i\omega\eta}(\varphi_{xx}\overline{h}_x + q\varphi_x\overline{h})dx \\ &= 2Re\int_0^T [(r'' + 2ivr' - v^2 r)(h_{1x} - ih_{2x} - ivh_1 - vh_2) + q(r' + ivr)(h_1 - ih_2)]dx \\ &= 2\int_0^T [(-\omega + 2|r| + 3v^2 + q)r'h_1 + v(v^2 - 3\omega + 3|r| + q)rh_2]dx. \end{aligned}$$

From (2.11), we have

$$\int_0^T qrh_2dx = -\int_0^T [(v^2-\omega+|r|)rh_2+2vr'h_1]dx$$

and replacing in the above equality, we obtain

$$\int_0^T [(-\omega+2|r|+v^2+q)r'h_1+v(2|r|-2\omega)rh_2]dx = 0.$$

Substituting $h_1 = \gamma_2 r'(\bar{x})+\vartheta$ in the above equality and using the orthogonality condition $\langle r',\vartheta\rangle = \langle r',\gamma_1\psi_0+\theta_1\rangle = 0$, we obtain

$$\gamma_2||r'||^2\left(-\omega+v^2+q+\frac{2||\sqrt{r}r'||^2}{||r'||^2}\right)+\int_0^T [v(2|r|-2\omega)rh_2+2|r|r'\vartheta]dx = 0.$$

As $\frac{2||\sqrt{r}r'||^2}{||r'||^2} \geq \omega$ (see estimate B1 from Appendix 1),

$$\begin{aligned}
|\gamma_2|\,||r'|| &\leq \frac{\left|\int_0^T [v(2|r|-2\omega)rh_2+2|r|r'\vartheta]dx\right|}{(v^2+q)||r'||}\\
&\leq 2\frac{|v|\,|||r|r-\omega r||\cdot||h_2||+2|||r|r'||\cdot||\vartheta||}{(v^2+q)||r'||}\\
&\leq m_0(||\vartheta||+||h_2||).
\end{aligned}$$

Hence

$$||h_1|| \leq |\gamma_2|\,||r'||+||\vartheta|| \leq (m_0+1)||\vartheta||+m_0||h_2||,$$

which yields

$$||\vartheta||^2 \geq \frac{||h_1||^2}{2(m_0+1)^2}-\left(\frac{m_0}{m_0+1}\right)^2||h_2||^2.$$

Replacing in (2.21), we finally obtain

$$M_1 \geq \frac{C_1\lambda_2}{C_2 2(m_0+1)^2}||h_1||^2-\frac{C_1\lambda_2 m_0^2}{C_2(m_0+1)^2}||h_2||^2-\frac{C_1\lambda_2 D_2+C_2D_1}{C_2}|h||^4. \qquad (2.22)$$

The estimate for ΔM.

From (2.13) and (2.22), we have (fixing q large and therefore m_0 small enough)

$$\begin{aligned}
M_1+M_2 &\geq \left(\tfrac{C_1\lambda_2-2C_2\alpha^2\epsilon_1 m_0^2}{2C_2(m_0+1)^2}\right)||h_1||^2+\left(\tfrac{C_2\alpha^2\epsilon_1-2C_1\lambda_2 m_0^2}{2C_2(m_0+1)^2}\right)||h_2||^2\\
&\quad -\tfrac{C_1\lambda_2D_2+C_2D_1}{C_2}||h||^4 \geq C_3||h||^2-D_3||h||^4.
\end{aligned}$$

On the other hand, estimating directly I_2 from below (for this purpose we use its initial formula), we obtain

$$\begin{aligned} I_2 &\geq ||h_x||^2 + \int_0^T (\omega + v^2 - 2|\varphi|)|h|^2 dx - 2|v| \int_0^T |h| \cdot |h_x| dx \\ &\geq ||h_x||^2 + (\omega + v^2 - 2\max|r|)||h||^2 - 2v^2||h||^2 - \tfrac{1}{2}||h_x||^2 \\ &= \tfrac{1}{2}||h_x||^2 + (\omega - v^2 - 2r_0)||h||^2. \end{aligned}$$

Let $0 < m < \frac{1}{2}$. We have

$$\begin{aligned} \Delta M &= 2mI_2 + (1-2m)(M_1 + M_2) + I_3 \\ &\geq m||h_x||^2 + 2m(\omega - v^2 - 2r_0)||h||^2 + (1-2m)(C_3||h||^2 - D_3||h||^4) + I_3 \\ &\geq m||h_x||^2 + [(2m(\omega - v^2 - 2r_0) + (1-2m)C_3]||h||^2 - (1-2m)D_3||h||^4 - |I_3|. \end{aligned}$$

We choose m, so that $2mq = (1-2m)C_3 + 2m(\omega - v^2 - 2r_0)$, i.e.

$$2m = \frac{C_3}{q + C_3 - \omega + v^2 + 2r_0}.$$

From the continuity of $|z|$ and $|z|e^{-2iargz}$, we have

$$|I_3| < \frac{mq}{2}||h||^2$$

From the inequality

$$|h|^2 \leq \frac{1}{T}\int_0^T |h|^2 dx + 2\left(\int_0^T |h|^2 dx \int_0^T |h_x|^2 dx\right)^{\frac{1}{2}}$$

we obtain

$$|h|^2 \leq \frac{1}{T}\int_0^T |h|^2 dx + \sqrt{q}\int_0^T |h|^2 dx + \frac{1}{\sqrt{q}}\int_0^T |h_x|^2 dx.$$

Hence for sufficiently large q, we obtain

$$\max|h(x,t)|^2 \leq \frac{2}{\sqrt{q}} d_q^2(u,\varphi)$$

and moreover $||h||^2 \leq q^{-1} d_q^2(u,\varphi)$. Consequently we can choose $\delta_0 > 0$, such that for $d_q(u,\varphi) < \delta_0$, we will have $[\max(4|a|^{\frac{1}{2}}|h| + |h|^2) + (1-2m)D_3|a|^{\frac{1}{2}}]||h||^2 \leq qm$.

Finally, we obtain that if $d_q(u,\varphi) < \delta_0$, then $\Delta M \geq m d_q^2(u,\varphi)$. Proposition 2.1 is completely proved. □

Proof of Theorem 2.1. We split the proof of our main result into two steps. We begin with the special case $P(u) = P(\varphi)$. Assume that m, q, δ_0 have been selected

according to Proposition 2.1. Since ΔM does not depend on $t, t \in [0, \infty)$, there exists a constant l such that $\Delta M \leq l d^2(u, \varphi)|_{t=0}$. Below, we shall assume without loss of generality that $l \geq 1, q \geq 1$.

Let

$$\varepsilon > 0, \quad \delta = \min\left(\left(\frac{m}{lq}\right)\frac{\delta_0}{2}, \left(\frac{m}{l}\right)^{1/2}\varepsilon\right)$$

and $d(u, \varphi)|_{t=0} < \delta$. Then

$$d_q(u, \varphi) \leq q^{1/2} d(u, \varphi)|_{t=0} < \frac{\delta_0}{2}$$

and Lemma 2.2 yields that there exists a $t_0 > 0$ such that $d_q(u, \varphi) < \delta_0$ if $t \in [0, t_0)$. Then, by virtue of Proposition 2.1 we have

$$\Delta M \geq m d_q^2(u, \varphi), \quad t \in [0, t_0).$$

Let t_{max} be the largest value such that

$$\Delta M \geq m d_q^2(u, \varphi), \quad t \in [0, t_{max}).$$

We assume that $t_{max} < \infty$. Then, for $t \in [0, t_{max}]$ we have

$$d_q^2(u, \varphi) \leq \frac{\Delta M}{m} \leq \frac{l}{m} d^2(u, \varphi)|_{t=0} < \frac{l}{m}\delta^2 \leq \frac{\delta_0^2}{4}.$$

Applying once again Lemma 2.2, we obtain that there exists $t_1 > t_{max}$ such that

$$d_q(u, \varphi) < \delta_0, \quad t \in [0, t_1).$$

By virtue of the proposition, this contradicts the assumption $t_{max} < \infty$. Consequently, $t_{max} = \infty$,

$$\Delta M \geq m d_q^2(u, \varphi) \geq m d^2(u, \varphi), \quad t \in [0, \infty).$$

Therefore,

$$d^2(u, \varphi) \leq \frac{\Delta M}{m} \leq \frac{l}{m}\delta^2 < \varepsilon^2, \quad t \in [0, \infty),$$

which proves the theorem in the special case.

Now we proceed to release the restriction $P(u) = ||u||^2 = ||\varphi||^2 = P(\varphi)$. We have

$$||\varphi|| = (16\alpha^3\sqrt{1 - k^2 + k^4}\left[(k^2 - 2 + \sqrt{1 - k^2 + k^4})K(k) + 3E(k)\right])^{1/2}.$$

Below, we are going to apply a perturbation argument, freezing for a while the period T and the parameters ω, c in (2.3). We claim there are respective parameter values ω^*, c^*, and corresponding φ^*, r^*, α^*, k^*, see (2.2), (2.3) and (2.5), such that φ^* has a period T in x and moreover, $||\varphi^*|| = ||u||$. By (2.9), we obtain the equations

$$\begin{aligned} &\frac{2K(k^*)}{\alpha^*} - T = 0, \\ &||r^*||^2 - ||u||^2 = 0. \end{aligned} \tag{2.23}$$

Moreover, one has $||\varphi^*|| = ||u||$ and we could use the restricted result we established above. As $k = k^*(T, ||\varphi||)$, $\alpha = \alpha^*(T, ||\varphi||)$, it remains to apply the implicit function theorem to (2.23). Since the corresponding Jacobian determinant reads

$$\begin{vmatrix} \frac{\partial}{\partial k^*}\left(\frac{2K(k^*)}{\alpha^*}\right) & \frac{\partial}{\partial \alpha^*}\left(\frac{2K(k^*)}{\alpha^*}\right) \\ \frac{\partial}{\partial k^*}||r^*||^2 & \frac{\partial}{\partial \alpha^*}||r^*||^2 \end{vmatrix} > 0$$

the needed properties are established.

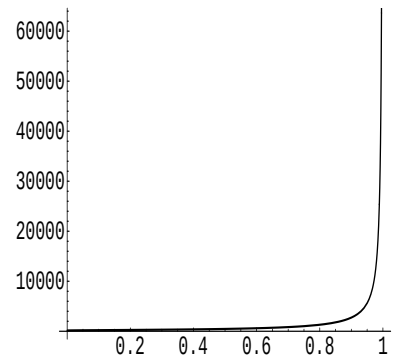

Figure 1: Graph of the Jacobian determinant

By (2.23) and our assumption, we have

$$\frac{K(k)}{\alpha} = \frac{K(k^*)}{\alpha^*} = \frac{T}{2}. \tag{2.24}$$

Next, choosing $\eta = 2(\omega^* - \omega)t, \;\; \xi = 0$, we obtain inequality

$$d^2(\varphi^*, \varphi) \leq (1 + v^2)||r^* - r||^2 + ||r^{*'} - r^{'}||^2.$$

Denote for while $\Phi(\rho) = 2\alpha^2(1 - 2k^2 + \sqrt{1 - k^2 + k^4} + 8k^2 cn^2(\alpha x; k))$, where $k = k(\rho)$ is determined from $K(k) = \frac{1}{2}\rho T$. Then $r^* - r = \Phi(\alpha^*) - \Phi(\alpha) = (\alpha^* - \alpha)\Phi'(\rho)$ with some appropriate ρ. Moreover, $|\Phi'(\rho)| \leq C_0$ with constant independent of the values with $*$ accent. Hence $|r^* - r| \leq C_0|\alpha^* - \alpha|$. Similarly $|r'^* - r'| \leq C_1|\alpha^* - \alpha|$. All this, together with (2.24) yields

$$d(\varphi^*, \varphi) \leq C|\alpha^* - \alpha|C = \frac{2C}{T}|K(k^*) - K(k)| = \frac{2C}{T}|K'(k)||k^* - k|. \tag{2.25}$$

Let $\varepsilon > 0$. From the inequalities

$$|||\varphi^*|| - ||\varphi||| = |||u|| - ||\varphi||| \leq d(u, \varphi)|_{t=0} < \delta$$

it follows that

$$-||r||\delta < (||\varphi||)^{-1}||\varphi^*|| - 1 < ||r||\delta$$

and, consequently, $|\,||r^*||^2 - ||r||^2\,| < ||r||^2\delta_1$, where $\delta_1 = (1+||r||\delta)^2 - 1$.

On the other hand, we have (using (2.24) again)

$$
\begin{aligned}
&|\,||r^*||^2 - ||r||^2\,| \\
&= \tfrac{2^{7/2}}{T^{3/2}}\Big|(K(k^*))^{3/2}\sqrt{1-k^{*2}+k^{*4}}\left[(k^{*2}-2+\sqrt{1-k^{*2}+k^{*4}}K(k^*)+3E(k^*)\right] \\
&-(K(k))^{3/2}\sqrt{1-k^2+k^4}\left[(k^2-2+\sqrt{1-k^2+k^4}K(k)+3E(k)\right]\Big| \\
&\geq C_2||k^*-k|,
\end{aligned}
\tag{2.26}
$$

Thus combining (2.25) and (2.26), we get

$$d(u,\varphi^*)|_{t=0} \leq d(u,\varphi)|_{t=0} + d(\varphi,\varphi^*)|_{t=0} < \delta + ||r||^2\widetilde{C}\delta_1 = \delta_0.$$

We select δ sufficiently small and apply the part of the theorem which has been already proved,

$$d(u,\varphi^*)|_{t=0} < \delta_0 \Rightarrow d(u,\varphi^*) < \frac{\varepsilon}{2}, \quad t\in[0,\infty).$$

Choosing an appropriate $\delta > 0$, we obtain that

$$d(u,\varphi) \leq d(u,\varphi^*) + d(\varphi,\varphi^*) < \varepsilon,$$

for all $t\in[0,\infty)$. Theorem 2.1 is completely proved. □

3 Existence and stability of periodic traveling waves for the cubic Schrödinger equation

Consider the cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2u = 0, \tag{3.1}$$

where $u = u(x,t)$ is a complex-valued function of $(x,t)\in\mathbb{R}^2$.

Equation (3.1) possesses the following family of traveling-wave solutions

$$\varphi(x,t) = e^{i(vx+(\omega-v^2)t)}r(x-2vt), \tag{3.2}$$

where ω and v are real parameters and the real-valued function $r(x)$ satisfies equation

$$r'' - \omega r + r^3 = 0. \tag{3.3}$$

Integrating once again, we obtain

$$r'^2 - \omega r^2 + \frac{1}{2}r^4 = c \tag{3.4}$$

and r is a periodic function provided that the energy level set $H(x,y)=c$ of the Hamiltonian system $dH=0$,

$$H(x,y)=y^2-\omega x^2+\frac{1}{2}x^4,$$

contains an oval (a simple closed real curve free of critical points). The level set $H(x,y)=c$ contains two periodic trajectories if $\omega>0$, $c\in(-\frac{1}{2}\omega^2,0)$ and a unique periodic trajectory if $\omega\in\mathbb{R}$, $c>0$. Under these conditions, the solution of (3.3) is determined by $H(r,r')=c$ and r is periodic of period $T=T(\omega,c)$.

Below, we are going to consider the case $c<0$. Let us denote by $r_0>r_1>0$ the positive roots of $\frac{1}{2}r^4-\omega r^2-c=0$. Then, up to a translation, we obtain the respective explicit formulas

$$r(z)=\mp r_0 dn(\alpha z;k),\quad k^2=\frac{r_0^2-r_1^2}{r_0^2}=\frac{-2\omega+2r_0^2}{r_0^2},\quad \alpha=\frac{r_0}{\sqrt{2}},\quad T=\frac{2K(k)}{\alpha}. \tag{3.5}$$

Recall that $K(k)$ and $E(k)$ are, as usual, the complete elliptic integrals of the first and second kind in a Legendre form. By (3.5), one also obtains $\omega=(2-k^2)\alpha^2$ and, finally,

$$T=\frac{2\sqrt{2-k^2}K(k)}{\sqrt{\omega}},\quad k\in(0,1),\quad T\in I=\left(\frac{2\pi}{\sqrt{\omega}},\infty\right). \tag{3.6}$$

We take $\omega>0$, an energy level $c\in(-\frac{1}{2}\omega^2,0)$ and let T be the (minimal) period of $r(x)$. Assume that $v\neq 0$ is chosen to satisfy $vT/2\pi\in\mathbb{Z}$. Then there are two periodic solutions r of (3.3): left ($r<0$) and right ($r>0$) and the corresponding traveling-wave solution $\varphi(x,t)$ is periodic with respect to x of period T.

Lemma 3.1. *For any $\omega>0$ and $T\in I$, there is a constant $c=c(\omega)$ such that the periodic traveling-wave solution* (3.5) *determined by $H(r,r')=c(\omega)$ has a period T. The function $c(\omega)$ is differentiable.*

Proof. See [10], Lemma 3.1.

Equation (3.1) has the following conservation laws

$$Q(u)=i\int_0^T \overline{u}_x u dx,\quad P(u)=\int_0^T |u|^2 dx,\quad \tilde{E}(u)=\int_0^T (|u_x|^2-\frac{|u|^4}{2})dx.$$

Let us consider the functional

$$\tilde{M}(u)=\tilde{E}(u)+(\omega+v^2)P(u)-2vQ(u).$$

Theorem 3.1. *Let φ be given by* (3.2)*, with $r\neq 0$. For each $\varepsilon>0$ there exists $\delta>0$ such that if $u(x,t)$ is a solution of* (3.1) *and $d(u,\varphi)_{|t=0}<\delta$, then $d(u,\varphi)<\varepsilon$ $\forall t\in[0,\infty)$.*

The crucial step in the proof will be to verify the following statement.

Proposition 3.1. *There exist positive constants m, q, δ_0 such that if u is a periodic solution of* (3.1)*, $u(x,t) = u(x+T,t)$, $P(u) = P(\varphi)$ and $d_q(u,\varphi) < \delta_0$, then*

$$\tilde{M}(u) - \tilde{M}(\varphi) \geq m d_q^2(u,\varphi). \tag{3.7}$$

Proof. In order to estimate $\Delta\tilde{M} = \tilde{M}(u) - \tilde{M}(\varphi)$, we set

$$u(x,t) = e^{i\eta}\varphi(x-\xi,t) + h(x,t),$$

$$\zeta = v(x-\xi) + (\omega + v^2)t + \eta,$$

and integrating by parts in the terms containing h_x and $\overline{h}_x$, we obtain

$$\begin{aligned}
\Delta\tilde{M} &= \tilde{M}(u) - \tilde{M}(\varphi) \\
&= 2Re\int_0^T e^{i\eta}[-\varphi_{xx} + (\omega + v^2 - |\varphi|^2)\varphi + 2iv\varphi_x]\overline{h}dx \\
&+ \int_0^T [|h_x|^2 + (\omega + v^2 - 2|\varphi|^2)|h|^2 - |\varphi|^2 Re(e^{-2i\zeta}h^2) - 2ivh\overline{h}_x]dx \\
&- \tfrac{1}{2}\int_0^T |h|^2(4Re(e^{i\eta}\varphi\overline{h}) + |h|^2)dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Using that $r(x)$ satisfies the equation (3.3) we obtain that $I_1 = 0$.

Let

$$h(x,t) = (h_1 + ih_2)e^{i\zeta},$$

where h_1 and h_2 are real periodic functions with period T. Using (2.10), for I_2 we obtain

$$I_2 = \int_0^T [h_{1x}^2 + (\omega - 3r^2)h_1^2]dx + \int_0^T [h_{2x}^2 + (\omega - r^2)h_2^2]dx = \tilde{M}_1 + \tilde{M}_2$$

Introduce in $L^2[0,T]$ the self-adjoint operators L_1 and L_2 generated by the differential expressions

$$\begin{aligned}
L_1 &= -\partial_x^2 + (\omega - 3r^2), \\
L_2 &= -\partial_x^2 + (\omega - r^2),
\end{aligned} \tag{3.8}$$

with periodic boundary conditions in $[0,T]$.

We use now (3.5) and (3.6) to rewrite operators L_1, L_2 in more appropriate form. From the expression for $r(x)$ from (3.5) and the relations between elliptic functions $sn(x)$, $cn(x)$ and $dn(x)$, we obtain

$$L_1 = \alpha^2[-\partial_y^2 + 6k^2 sn^2(y) - 4 - k^2]$$

where $y = \alpha x$.

It is well-known that the first five eigenvalues of $\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0, 4K(k)]$, where $K(k)$ is the complete elliptic integral of the first kind, are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned}
&\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, && \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k),\\
&\nu_1 = 1 + k^2, && \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k),\\
&\nu_2 = 1 + 4k^2, && \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k),\\
&\nu_3 = 4 + k^2, && \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k),\\
&\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, && \phi_4(y) = 1 - (1 + k^2 + \sqrt{1-k^2+k^4})sn^2(y,k).
\end{aligned}$$

It follows that the first three eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0, 2K(k)]$ (that is, in the case of left and right family), are simple and $\lambda_0 = \alpha^2(\nu_0 - \nu_3) < 0,\ \lambda_1 = \alpha^2(\nu_3 - \nu_3) = 0,\ \lambda_2 = \alpha^2(\nu_4 - \nu_3) > 0$. The corresponding eigenfunctions are $\psi_0 = \phi_0(\alpha x), \psi_1 = r'(x), \psi_2 = \phi_4(\alpha x)$.

Similarly, for the operator L_2 we have

$$L_2 = \alpha^2[-\partial_y^2 + 2k^2 sn^2(y,k) - k^2]$$

in the case of left and right family. The spectrum of $\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y,k)$ is formed by bands $[k^2, 1] \cup [1 + k^2, +\infty)$. The first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on $[0, 4K(k)]$ are simple and

$$\begin{aligned}
&\epsilon_0 = k^2, && \theta_0(y) = dn(y,k),\\
&\epsilon_1 = 1, && \theta_1(y) = cn(y,k),\\
&\epsilon_2 = 1 + k^2, && \theta_2(y) = sn(y,k).
\end{aligned}$$

From (3.4) it follows that zero is an eigenvalue of L_2 and it is the first eigenvalue in the case of left and right family, with corresponding eigenfunction $r(x)$.

Estimates for $\tilde{M}_2$.

As in Section 2, we use below $r = r(\bar{x})$ with an argument $\bar{x} = x - \xi - 2vt$. From the explanations above, we know that when considered in $[0, T]$, the operator L_2 has an eigenfunction r corresponding to zero eigenvalue and the rest of the spectrum is contained in (α^2, ∞).

The derivative of $d_q^2(u, \varphi)$ with respect to η at the point where the minimum is attained is equal to zero. Together with (3.3), this yields

$$\int_0^T \left[\left(v^2 - \omega + q + r^2\right) r h_2 + 2vr'h_1\right] dx = 0 \tag{3.9}$$

We set $h_2 = \beta r(\bar{x}) + \theta$, $\int_0^T \theta r dx = 0$. Substituting in (3.9), we obtain

$$\beta ||r||^2 \left[v^2 - \omega + q + \frac{||r^2||^2}{||r||^2} \right] + \int_0^T [\theta r^3 + 2vr'h_1] dx = 0.$$

Using that $\frac{2||r^2||^2}{||r||^2} \geq w$ (see estimate A2 of the Appendix2), we obtain the estimate

$$\begin{aligned} |\beta| \, ||r|| &\leq \frac{\left| \int_0^T (\theta r^3 + 2vr'h_1) dx \right|}{(q+v^2)||r||} \\ &\leq \frac{||r^3|| \cdot ||\theta|| + 2|v| \, ||r'|| \cdot ||h_1||}{(q+v^2)||r||} \\ &\leq m_0(||\theta|| + ||h_1||), \end{aligned}$$

where $m_0 = 2m_1(v,\omega)/(q+\omega^2)$ and

$$m_1(v,\omega) = \max_{c \in [-\frac{1}{2}\omega^2, 0]} \left(\frac{||r^3||}{||r||}, \frac{2|v| \, ||r'||}{||r||}, \frac{2|v| \, ||r^3 - \omega r||}{||r'||}, \frac{3||r^2 r'||}{||r'||} \right)$$

(the third and fourth item are included for later use). It is obvious that the first and the last fractions are bounded. For the second and the third ones, see estimates D2 in Appendix 2. We will use below that for v and ω fixed, $m_0 \to 0$ when $q \to \infty$. Further,

$$||h_2|| \leq |\beta| \, ||r|| + ||\theta|| \leq m_0(||\theta|| + ||h_1||) + ||\theta|| = (m_0+1)||\theta|| + m_0||h_1||.$$

Hence, we obtain

$$||\theta||^2 \geq \frac{||h_2||^2}{2(m_0+1)^2} - \left(\frac{m_0}{m_0+1} \right)^2 ||h_1||^2. \tag{3.10}$$

Since $L_2 r = 0$ and $\langle \theta, r \rangle = 0$, then from the spectral properties of the operator L_2, it follows

$$M_2 = \langle L_2 h_2, h_2 \rangle = \langle L_2 \theta, \theta \rangle \geq \alpha^2 \langle \theta, \theta \rangle \geq \frac{\omega}{2} ||\theta||^2.$$

From here and (3.10), one obtains

$$M_2 \geq \frac{\omega}{4(m_0+1)^2} ||h_2||^2 - \frac{\omega m_0^2}{2(m_0+1)^2} ||h_1||^2. \tag{3.11}$$

Estimates for $\tilde{M}_1$.

We set

$$h_1 = \gamma_1 \psi_0(\bar{x}) + \gamma_2 r'(\bar{x}) + \theta_1, \quad r(\bar{x}) = \nu \psi_0(\bar{x}) + \psi, \tag{3.12}$$

where

$$\langle \theta_1, \psi_0 \rangle = \langle \theta_1, r' \rangle = \langle \psi, \psi_0 \rangle = \langle \psi_0, r' \rangle = \langle \psi, r' \rangle = 0 \tag{3.13}$$

and γ_1, γ_2 and ν are some constants. As the same line as for M_1, we obtain

$$\gamma_1^2||\psi_0||^2 = \frac{1}{\nu^2||\psi_0||^2}\left(\frac{1}{2}||h||^2 + \int_0^T \psi\theta_1 dx\right)^2. \tag{3.14}$$

and

$$\begin{aligned}\tilde{M}_1 &\geq \left(\lambda_2 + \lambda_0(1+\tfrac{1}{d})\tfrac{||\psi||^2}{\nu^2||\psi_0||^2}\right)||\theta_1||^2 + \tfrac{\lambda_0(1+d)}{4\nu^2||\psi_0||^2}||h||^4 \\ &\geq C_1\lambda_2||\theta_1||^2 - D_1 w||h||^4.\end{aligned} \tag{3.15}$$

(see the estimates in point C2 of the Appendix2).

We denote $\vartheta = h_1 - \gamma_2 r'(\bar{x}) = \gamma_1\psi_0(\bar{x}) + \theta_1$. Then from (3.14), (3.15) and (since $\lambda_2 = -\omega + \sqrt{4\omega^2 + 6c}$ and $\lambda_2 = -\omega - \sqrt{4\omega^2 + 6c}$ the inequalities $\lambda_2 \leq \frac{1}{3}|\lambda_0| \leq w$, we have

$$\begin{aligned}||\vartheta||^2 = \gamma_1^2||\psi_0||^2 + ||\theta_1||^2 &\leq \left(1 + \tfrac{(d+1)||\psi||^2}{d\nu^2||\psi_0||^2}\right)||\theta_1||^2 + \tfrac{1+d}{4\nu^2||\psi_0||^2}||h||^4 \\ &\leq C_2||\theta_1||^2 + D_2\omega^{-\frac{1}{2}}||h||^4.\end{aligned}$$

Then

$$||\theta_1||^2 \geq \frac{||\vartheta||^2}{C_2} - \frac{D_2||h||^4}{C_2\omega^{\frac{1}{2}}}$$

and hence, by (3.15) and $\lambda_2 \leq w$,

$$\begin{aligned}M_1 &\geq \frac{C_1\lambda_2}{C_2}||\vartheta||^2 - \frac{C_1D_2 + C_2D_1}{C_2}\omega^{\frac{1}{2}}||h||^4 \\ &= C_3\lambda_2||\vartheta||^2 - D_3\omega^{\frac{1}{2}}||h||^4.\end{aligned} \tag{3.16}$$

After differentiating (2.8) with respect to ξ and using (3.3), we obtain

$$\begin{aligned}0 &= 2Re\int_0^T e^{i\eta}(\varphi_{xx}\overline{h}_x + q\varphi_x\overline{h})dx \\ &= 2\int_0^T [(-\omega + 3v^2 + 3r^2 + q)r'h_1 + v(-3\omega + v^2 + 3r^2 + q)rh_2]dx.\end{aligned}$$

From (3.9), we have

$$\int_0^T qrh_2 dx = -\int_0^T [2vr'h_1 + (-\omega + v^2 + r^2)rh_2]dx$$

and replacing in the above equality, we obtain

$$\int_0^T [(-\omega + v^2 + 3r^2 + q)r'h_1 + v(-2\omega + 2r^2)rh_2]dx = 0.$$

Substituting $h_1 = \gamma_2 r'(\bar{x}) + \vartheta$ in the above equality and using the orthogonality condition $\langle r', \vartheta\rangle = \langle r', \gamma_1\psi_0 + \theta_1\rangle = 0$, we obtain

$$\gamma_2||r'||^2\left(-\omega + v^2 + q + \frac{3||rr'||^2}{||r'||^2}\right) + \int_0^T [2v(-\omega + r^2)rh_2 + 3r^2r'\vartheta]dx = 0.$$

Using that $\frac{3||rr'||^2}{||r'||^2} \geq \omega$ (see the estimates in point B2 of the Appendix2), we further have

$$\begin{aligned} |\gamma_2|\,||r'|| &\leq \frac{\left|\int_0^T [2v(-\omega + r^2)rh_2 + 3r^2r'\vartheta]dx\right|}{(v^2+q)||r'||} \\ &\leq \frac{2|v|\,||-\omega r + r^3|| \cdot ||h_2|| + 3||r^2r'|| \cdot ||\vartheta||}{(v^2+q)||r'||} \\ &\leq m_0(||\vartheta|| + ||h_2||). \end{aligned}$$

Hence

$$||h_1|| \leq |\gamma_2|\,||r'|| + ||\vartheta|| \leq (m_0+1)||\vartheta|| + m_0||h_2||,$$

which yields

$$||\vartheta||^2 \geq \frac{||h_1||^2}{2(m_0+1)^2} - \left(\frac{m_0}{m_0+1}\right)^2 ||h_2||^2.$$

Replacing in (3.16), we finally obtain

$$M_1 \geq \frac{C_3\lambda_2}{2(m_0+1)^2}||h_1||^2 - \frac{C_3\lambda_2 m_0^2}{(m_0+1)^2}||h_2||^2 - D_3\omega^{\frac{1}{2}}|h||^4. \tag{3.17}$$

The estimate for ΔM.

From (3.11) and (3.17), one obtains

$$M_1 + M_2 \geq \frac{C_3\lambda_2 - \omega m_0^2}{2(m_0+1)^2}||h_1||^2 + \frac{\omega - 4C_3\lambda_2 m_0^2}{4(m_0+1)^2}||h_2||^2 - D_3\omega^{\frac{1}{2}}||h||^4.$$

We now fix q so that $\omega m_0^2 \leq \frac{1}{2}C_3\lambda_2$ and assuming that $C_3 \leq \frac{1}{2}$ (which is no loss of generality), one has also $4C_3\lambda_2 m_0^2 \leq \frac{1}{2}\omega$. Therefore we come to

$$M_1 + M_2 \geq C_4\lambda_2(||h_1||^2 + ||h_2||^2) - D_3\omega^{\frac{1}{2}}||h||^4 = C_4\lambda_2||h||^2 - D_3\omega^{\frac{1}{2}}||h||^4$$

where C_4 and D_3 are absolute positive constants independent on the parameters of the system.

On the other hand, estimating directly I_2 from below (for this purpose we use its initial formula), we have

$$\begin{aligned} I_2 &\geq ||h_x||^2 + \int_0^T (\omega + v^2 - 2r^2)|h|^2dx - 2|v|\int_0^T |h| \cdot |h_x|dx - \int_0^T r^2|h|^2dx \\ &\geq ||h_x||^2 + (\omega + v^2 - 2r_0^2)||h||^2 - 2v^2||h||^2 - \tfrac{1}{2}||h_x||^2 - r_0^2||h||^2 \\ &= \tfrac{1}{2}||h_x||^2 - (v^2 + 5\omega)||h||^2. \end{aligned}$$

Similarly, $|I_3| \le \max(4|2\omega|^{\frac{1}{2}}|h| + |h|^2)||h||^2$. Let $0 < m < \frac{1}{2}$. We obtain

$$\begin{aligned}
\Delta M &= 2mI_2 + (1-2m)(M_1 + M_2) + I_3 \\
&\ge m||h_x||^2 - 2m(v^2 + 5\omega)||h||^2 + (1-2m)(C_4\lambda_2||h||^2 - D_3\omega^{\frac{1}{2}}||h||^4) \\
&- \max(4|2\omega|^{\frac{1}{2}}|h| + |h|^2)||h||^2 \\
&= m||h_x||^2 + [-2m(v^2 + 5\omega) + (1-2m)C_4\lambda_2]\,||h||^2 \\
&-[\max(4|2\omega|^{\frac{1}{2}}|h| + |h|^2) + (1-2m)D_3\omega^{\frac{1}{2}}||h||^2]||h||^2.
\end{aligned}$$

We choose m, so that

$$2qm = (1-2m)C_4\lambda_2 - 2m(v^2 + 5\omega), \quad \text{i.e. } 2m = \frac{C_4\lambda_2}{q + v^2 + 5\omega + C_4\lambda_2} < 1.$$

From the inequality

$$|h|^2 \le \frac{1}{T}\int_0^T |h|^2 dx + 2\left(\int_0^T |h|^2 dx \int_0^T |h_x|^2 dx\right)^{\frac{1}{2}}$$

we obtain

$$|h|^2 \le \frac{1}{T}\int_0^T |h|^2 dx + \sqrt{q}\int_0^T |h|^2 dx + \frac{1}{\sqrt{q}}\int_0^T |h_x|^2 dx.$$

Hence for sufficiently large q, we obtain

$$\max|h(x,t)|^2 \le \frac{2}{\sqrt{q}} d_q^2(u,\varphi)$$

and moreover $||h||^2 \le q^{-1}d_q^2(u,\varphi)$. Consequently we can choose $\delta_0 > 0$, such that for $d_q(u,\varphi) < \delta_0$, we will have $[\max(4|2\omega|^{\frac{1}{2}}|h| + |h|^2) + (1-2m)D_3\omega^{\frac{1}{2}}]||h||^2 \le qm$.

Finally, we obtain that if $d_q(u,\varphi) < \delta_0$, then $\Delta M \ge m d_q^2(u,\varphi)$. Proposition 3.1 is completely proved. □

Proof of Theorem 3.1. The proof of the theorem in the case $P(u) = ||u||^2 = ||\varphi||^2 = P(\varphi)$ is the same as in Theorem 2.1. If $P(u) \ne P(\varphi)$, we proceed similarly as in Theorem 2.1. We have $||\varphi|| = (2\sqrt{2}r_0E(k))^{1/2}$, where r_0 is given by (3.5).

We claim there are respective parameter values ω^*, c^*, and corresponding φ^*, r^*, r_0^*, k^*, see (3.5) and (3.6), such that φ^* has a period T in x and moreover, $2\sqrt{2}r_0^*E(k^*) = ||u||^2$. By (3.5), we obtain the equations

$$\begin{aligned}
&\frac{2\sqrt{2}K(k^*)}{r_0^*} - T = 0, \\
&2\sqrt{2}r_0^*E(k^*) - ||u||^2 = 0.
\end{aligned} \tag{3.18}$$

If (3.18) has a solution $k^* = k^*(T, ||u||)$, $r_0^* = r_0^*(T, ||u||)$, then the parameter values we need are given by

$$2w^* = (2 - k^{*2})r_0^{*2}, \quad 2c^* = (k^{*2} - 1)r_0^{*4}.$$

Moreover, one has $||\varphi^*|| = ||u||$ and we could use the restricted result we established above. As $k = k^*(T, ||\varphi||)$, $r_0 = r_0^*(T, ||\varphi||)$, it remains to apply the implicit function theorem to (3.18). Since the corresponding Jacobian determinant reads

$$\begin{vmatrix} \frac{2\sqrt{2}K'(k^*)}{r_0^*} & -\frac{2\sqrt{2}K(k^*)}{{r_0^*}^2} \\ 2\sqrt{2}r_0^*E'(k^*) & 2\sqrt{2}E(k^*) \end{vmatrix} = \frac{8}{r_0^*}(KE)' > 0$$

the existence of ω^* and c^* with the needed properties is established.

By (3.18) and our assumption, we have

$$\frac{2\sqrt{2}K(k^*)}{r_0^*} = \frac{2\sqrt{2}K(k)}{r_0} = T. \tag{3.19}$$

Choosing $\eta = 2(\omega^* - \omega)t$, $\xi = 0$, we obtain the inequality

$$d^2(\varphi^*, \varphi) \leq (1 + v^2)||r^* - r||^2 + ||r^{*'} - r^{'}||^2$$

For $\Phi(\rho) = \rho dn(z\rho, k(\rho))$, where $k = k(\rho)$ is determined from $K(k) = \frac{1}{2}\rho T$, we get $r^* - r = \Phi(r_0^*) - \Phi(r_0) = (r_0^* - r_0)\Phi'(\rho)$ with some appropriate ρ. Moreover, $|\Phi'(\rho)| \leq C_0$. Hence $|r^* - r| \leq C_0|r_0^* - r_0|$, $|r^{*'} - r'| \leq C_1|r_0^* - r_0|$, and by (3.19)

$$d(\varphi^*, \varphi) \leq |r_0^* - r_0| \leq \frac{2\sqrt{2}C}{T}|K(k^*) - K(k)| \leq \frac{2\sqrt{2}C}{T}|K'(k)||k^* - k|.$$

From the inequalities

$$|||\varphi^*|| - ||\varphi||| = |||u|| - ||\varphi||| \leq d(u, \varphi)|_{t=0} < \delta$$

it follows that

$$-\left(2\sqrt{2}r_0E\right)^{-1/2}\delta < (||\varphi||)^{-1}||\varphi^*|| - 1 < \left(2\sqrt{2}r_0E\right)^{-1/2}\delta$$

and, therefore, $1 - \delta_1 < \frac{r_0^*E(\kappa^*)|}{r_0E} < 1 + \delta_1$, i.e. $|r_0^*E(\kappa^*) - r_0E| < r_0E\delta_1$, where $\delta_1 = (1 + (2r_0E)^{-1/2}\delta)^2 - 1$. On the other hand

$$|r_0^*E(k^*) - r_0E(k)| = \frac{2\sqrt{2}}{T}|K(k^*)E(k^*) - K(k)E(k)| = \frac{2\sqrt{2}}{T}|(KE)'(k)||k^* - k| \geq C_2|k^* - k|,$$

with appropriate $C_2 > 0$. Thus

$$d(u, \varphi^*)|_{t=0} \leq d(u, \varphi)|_{t=0} + d(\varphi, \varphi^*)|_{t=0} < \delta + r_0E\widetilde{C}\delta_1 = \delta_0.$$

The rest of the proof is the same as in Theorem 2.1. □

4 Appendix 1

Below, we provide some estimates needed in our proofs concerning the quadratic Schrödinger equation. Without loss of generality, we will assume that $r > 0$. The case of negative r is dealt with by changing $r \to -r$ in all equations from Section 1.

For $n \in \mathbb{Z}$ and $c \in (-\frac{1}{3}\omega^3, 0)$, consider the line integrals $I_n(c)$ and their derivatives $I_n'(c)$ given by

$$I_n(c) = \oint_{H=c} x^n y dx, \qquad I_n'(c) = \oint_{H=c} \frac{x^n dx}{2y} \tag{4.1}$$

where the integration is along the right oval contained in the level set $\{H = c\}$ and $H(x, y) = y^2 - \omega x^2 + \frac{2}{3}x^3$. These integrals would be useful because

$$\int_0^T r^n(t)dt = 2\int_0^{\frac{1}{2}T} r^n(t)dt = 2\int_{r_1}^{r_0} \frac{x^n dx}{\sqrt{c + \omega x^2 - \frac{2}{3}x^3}} = \oint_{H=c} \frac{x^n dx}{y} = 2I_n'(c). \tag{4.2}$$

(we applied a change of the variable $r(t) = x$ in the integral and used equation (2.3)). The properties of I_n are well known, see e.g. [9] for a similar treatment. Below, we list some facts we are going to use.

Lemma. (i) *The following identity holds:*

$$3ncI_{n-1} + 3\omega(n+3)I_{n+1} - (2n+9)I_{n+2} = 0, \quad n \in \mathbb{Z}$$

which implies

$$I_2 = \omega I_1, \quad I_3 = \tfrac{3}{11}cI_0 + \tfrac{12}{11}\omega^2 I_1, \quad I_4 = \tfrac{45}{143}c\omega I_0 + (\tfrac{6}{13}c + \tfrac{180}{143}\omega^3)I_1, \tag{4.3}$$

(ii) *The integrals I_0 and I_1 satisfy the system*

$$\begin{aligned} &6cI_0' + 2\omega^2 I_1' = 5I_0, \\ &6\omega c I_0' + (30c + 12\omega^3)I_1' = 35I_1. \end{aligned}$$

(iii) *The ratio $R(c) = I_1'(c)/I_0'(c)$ satisfies the Riccati equation and related system*

$$6c(3c + \omega^3)R'(c) = \omega^2 R^2(c) + 6cR(c) - 3\omega c, \qquad \begin{aligned} &\dot{c} = 6c(3c + \omega^3), \\ &\dot{R} = \omega^2 R^2 + 6cR - 3\omega c, \end{aligned} \tag{4.4}$$

which imply estimates

$$-\frac{3c}{\omega^2} \le R(c) \le \frac{5\omega}{6} - \frac{c}{2\omega^2}. \tag{4.5}$$

The equations in (i)–(iii) are derived in a standard way, see [9] for more details. The estimates (4.5) follow from the fact that, in the (c, R)-plane, the graph of $R(c)$ coincides with the concave separatrix trajectory of the system (4.4) contained in the

triangle with vertices $(0,0)$, $(-\frac{1}{3}\omega^3,\omega)$ and $(0,\frac{5\omega}{6})$ and connecting the first two of them.

After this preparation, we turn to prove the estimates we used in the preceding Section 2.

A1. The estimate for $A_1 = \frac{||r^{3/2}||^2}{||r||^2}$. By (4.2), (4.3) and the first inequality in (4.5), we have

$$A_1 = \frac{\int_0^T r^3 dt}{\int_0^T r^2 dt} = \frac{I_3'}{I_2'} = \frac{3}{5}\frac{cI_0' + 2\omega^2 I_1'}{\omega I_1'} = \frac{3c}{5\omega R} + \frac{6\omega}{5} \geq \omega.$$

B1. The estimate for $B_1 = \frac{2||r^{1/2}r'||^2}{||r'||^2}$. By (2.3) and (4.3), we have as above

$$B_1 = \frac{2\int_0^T r(c+\omega r^2 - \frac{2}{3}r^3)dt}{\int_0^T (c+\omega r^2 - \frac{2}{3}r^3)dt} = \frac{2(cI_1' + \omega I_3' - \frac{2}{3}I_4')}{cI_0' + \omega I_2' - \frac{2}{3}I_3'}$$

$$= \frac{6}{7}\frac{c\omega I_0' + (5c+2\omega^3)I_1'}{3cI_0' + \omega^2 I_1'} = \frac{6}{7}\frac{c\omega + (5c+2\omega^3)R}{3c+\omega^2 R} \geq \frac{12}{7}\omega.$$

To obtain the last inequality, we used both estimates in (4.5).

C1. The estimate for $C_1 = \lambda_2 + \lambda_0(1+\frac{1}{d})\frac{||\psi||^2}{\nu^2||\psi_0||^2}$. By (2.14) and (2.15) we have

$$||\psi||^2 = ||r||^2 - \nu^2||\psi_0||^2, \quad \text{where } \nu = \frac{\langle r, \psi_0\rangle}{||\psi_0||^2}.$$

Therefore

$$C_1 = \lambda_2 + \lambda_0\left(1+\frac{1}{d}\right)\left(\frac{||r||^2||\psi_0||^2}{\langle r,\psi_0\rangle^2} - 1\right).$$

Below, we need to use the following well-known equalities (see e.g. [7]) which are written out here for reader's convenience:

(i) $\int_0^K dn^2 x dx = E(k)$

(ii) $\int_0^K dn^4 x dx = \frac{4-2k^2}{3}E(k) + \frac{k^2-1}{3}K(k)$

(iii) $\int_0^K dn^6 x dx = \frac{8k^4-23k^2+23}{15}E(k) + \frac{4(k^2-1)(2-k^2)}{15}K(k)$

(iv) $\int_0^K dn^2 x sn^2 x dx = \frac{2k^2-1}{3k^2}E(k) + \frac{1-k^2}{3k^2}K(k)$

(iv) $\int_0^K dn^2 x sn^4 x dx = \frac{8k^4-3k^2-2}{15k^4}E(k) + \frac{2(1-k^2)(1+2k^2)}{15k^4}K(k)$

(v) $\int_0^{2K} cn^2 x dx = \frac{2}{k^2}[E(k) + (k^2-1)K(k)]$

(vi) $\int_0^{2K} sn^2 x dx = \frac{2}{k^2}[K(k) - E(k)]$

(vii) $\int_0^K dnxdx = \int_0^{2K} dnxsn^2xdx = \int_0^{2K} dnxcn^2xdx = \pi/2$

(viii) $\int_0^{2K} dnxsn^2xcn^2xdx = \pi/8$

(ix) $\int_0^{2\pi} dn^3xdx = \frac{2-k^2}{2}\pi$

Now, we first recall that

$$r(x) = \alpha^2[1 + k^2 + \sqrt{1 - k^2 + k^4} - 3k^2sn^2(\alpha x; k)],$$
$$\psi_0(x) = dn(x; k)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4})sn^2(x; k)] > 0.$$

From (2.2), we obtain after integration

$$||r||^2 = w\int_0^T r(x)dx = wr_1T + w(r_0 - r_1)\int_0^T cn^2(\alpha x)dx.$$

Then, by using (v) and the expressions of ω, r_1 etc, we calculate

$$||r||^2 = 16\alpha^3\sqrt{1 - k^2 + k^4}[(k^2 - 2 + \sqrt{1 - k^2 + k^4})K(k) + 3E(k)]$$

Similarly, by direct calculations and making use of (i)-(ix), we come to the expressions

$$||\psi_0||^2 = \frac{4}{\alpha}\frac{\sqrt{1 - k^2 + 4k^4}}{15k^4}\{(k^2 - 1)[(8k^4 + 3k^2 + 2) - 2(2k^2 + 1)\sqrt{1 - k^2 + 4k^4}]K(k)$$
$$+[(8k^4 - 3k^2 - 2)\sqrt{1 - k^2 + 4k^4} - 2(8k^6 - 4k^4 - k^2 - 1)]E(k)\}.$$

$$\langle r, \psi_0\rangle = \pi\alpha[(1 + k^2 + \sqrt{1 - k^2 + k^4})(1 - 2k^2 + \sqrt{1 - k^2 + 4k^4})$$
$$-\frac{9k^2}{4}(\frac{1}{3} - 2k^2 + \sqrt{1 - k^2 + 4k^4})]$$

Next, calculating the asymptotical expansions near $k = 0$, we obtain

$$||r||^2 = 16\pi\alpha^3(1 - \tfrac{3}{4}k^2 + \tfrac{39}{64}k^4 + \ldots)$$
$$\langle r, \psi_0\rangle = 4\pi\alpha(1 - \tfrac{7}{4}k^2 + \tfrac{71}{32}k^4 + \ldots)$$
$$||\psi_0||^2 = \frac{\pi}{\alpha}\sqrt{1 - k^2 + 4k^4}(1 - \tfrac{9}{4}k^2 + \tfrac{135}{64}k^4 + \ldots)$$
$$\frac{-\lambda_0}{\lambda_2 - \lambda_0} = \frac{1 - \frac{1}{2}k^2 + \frac{15}{16}k^4 + \ldots}{\sqrt{1 - k^2 + 4k^4}}$$

and, finally,

$$\frac{-\lambda_0}{\lambda_2 - \lambda_0}\frac{||r||^2||\psi_0||^2}{\langle r, \psi_0\rangle^2} = 1 - \frac{21}{32}k^4 + O(k^6). \tag{4.6}$$

Let us denote by $1-\delta$ the right-hand side of (4.6). Clearly, $\delta = \delta(k)$ satisfies $1 > \delta > 0$ for k small enough, say $0 < k \le k_0$. Fixing such a small k, then (4.6) yields

$$\frac{||r||^2||\psi_0||^2}{\langle r, \psi_0\rangle^2} = (1 - \delta)\left(1 - \frac{\lambda_2}{\lambda_0}\right) \le 1 + \frac{\lambda_2}{\lambda_0}(\delta - 1).$$

Therefore

$$C_1 = \lambda_2 + \lambda_0 \left(1 + \frac{1}{d}\right) \left(\frac{||r||^2||\psi_0||^2}{\langle r, \psi_0\rangle^2} - 1\right) \geq \lambda_2 \left[1 + \left(1 + \frac{1}{d}\right)(\delta - 1)\right] = C\lambda_2$$

where C is a positive constant, provided that d is chosen sufficiently large. Note that the above estimate is not uniform in k when k tends to zero. This is because $\delta = O(k^4)$ and hence $d = O(k^{-4})$.

When $k \geq k_0$, we can simply draw the graph of the corresponding function $f(k) = \frac{||r||^2||\psi_0||^2}{\langle r,\psi_0\rangle^2} - 1 + \frac{\lambda_2}{\lambda_0}$ to see that it is negative and placed far from zero. Hence, $C_1 \geq C\lambda_2$, too (uniformly for $k \geq k_0$).

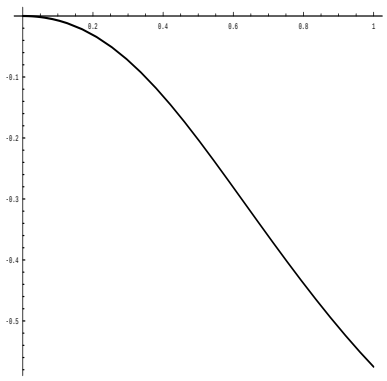

Figure 2: Graph of $f(k) < 0$

D1. The estimates for $D_{11} = \frac{||r^2 - \omega r||}{||r'||}$ and $D_{12} = \frac{||r'||}{||r||}$. Making use of statements (i) and (ii) of the Lemma, we have

$$D_{11}^2 = \frac{\int_0^T (r^2 - \omega r)^2 dt}{\int_0^T (c + \omega r^2 - \frac{2}{3}r^3) dt} = \frac{\omega^2 I_2' - 2\omega I_3' + I_4'}{cI_0' + \omega I_2' - \frac{2}{3}I_3'}$$

$$= \frac{5}{7} . \frac{-3c\omega I_0' + (6c + \omega^3) I_1'}{3cI_0' + \omega^2 I_1'} = \frac{-\omega I_0 + 2I_1}{I_0} \leq 2\omega.$$

The last inequality follows from the fact that the periodic trajectories of the Hamiltonian system lie inside the saddle loop, in the domain $H(x, y) < 0$ which implies $y^2 < x^2(\omega - \frac{2}{3}x)$ and $x < \frac{3}{2}\omega$.

Similarly,

$$D_{12}^2 = \frac{\int_0^T (c + \omega r^2 - \frac{2}{3}r^3) dt}{\int_0^T r^2 dt} = \frac{cI_0' + \omega I_2' - \frac{2}{3}I_3'}{I_2'}$$

$$= \frac{3cI_0' + \omega^2 I_1'}{5\omega I_1'} = \frac{(3c + \omega^3) I_0}{\omega(7I_1 - \omega I_0)} \leq \frac{3c + \omega^3}{5\omega^2}.$$

The last inequality is due to the known fact [14] (Theorem 12), that the function $Q(c) = I_1(c)/I_0(c)$ is strictly decreasing, therefore $Q(c) > Q(0) = \frac{6}{7}\omega$, the value obtained by direct calculation of elementary integrals.

5 Appendix 2

First of all, let us mention that the operator L_1, defined by differential expression (3.8) and equipped with periodic boundary conditions in $[0, T]$, has the following spectral data

$$\begin{aligned} &\lambda_0 = -\omega - \sqrt{4\omega^2 + 6c}, && \psi_0 = 3r^2 - 3\omega - \lambda_0, \\ &\lambda_1 = 0, && \psi_1 = r', \\ &\lambda_2 = -\omega + \sqrt{4\omega^2 + 6c}, && \psi_2 = 3r^2 - 3\omega - \lambda_2. \end{aligned} \tag{5.1}$$

For $n \in \mathbb{Z}$ and $c \in (-\frac{1}{2}\omega^2, 0)$, consider the line integrals $I_n(c)$ and their derivatives $I_n'(c)$ given by

$$I_n(c) = \oint_{H=c} x^n y dx, \qquad I_n(c) = \oint_{H=c} \frac{x^n dx}{2y} \tag{5.2}$$

where one can assume for definiteness that the integration is along the right oval contained in the level set $\{H = c\}$, $H(x, y) = y^2 - \omega x^2 + \frac{1}{2}x^4$. As above, these integrals satisfy $I_n'(c) = \frac{1}{2}\int_0^T r^n(z)dz$. The properties of I_n are well known. We only list some facts we are going to use.

Lemma. (i) *The following identity holds:*

$$(n + 6)I_{n+3} - 2\omega(n + 3)I_{n+1} - 2ncI_{n-1} = 0, \quad n \in \mathbb{Z}$$

which implies

$$I_3 = \omega I_1, \quad I_4 = \frac{2c}{7}I_0 + \frac{8\omega}{7}I_2, \quad I_6 = \frac{8\omega c}{21}I_0 + \left(\frac{2c}{3} + \frac{32\omega^2}{21}\right) I_2. \tag{5.3}$$

(ii) *The integrals I_0 and I_2 satisfy the system*

$$\begin{aligned} &4cI_0' + 2\omega I_2' = 3I_0, \\ &4\omega cI_0' + (12c + 8\omega^2)I_2' = 15I_2. \end{aligned}$$

(iii) *The ratio $R(c) = I_2'(c)/I_0'(c)$ satisfies the Riccati equation and related system*

$$(8c^2 + 4\omega^2 c)R'(c) = -2\omega c + 4cR(c) + \omega R^2(c), \qquad \begin{aligned} &\dot{c} = 8c^2 + 4\omega^2 c, \\ &\dot{R} = -2\omega c + 4cR + \omega R^2, \end{aligned} \tag{5.4}$$

which imply estimates

$$-\frac{2c}{\omega} \le R(c) \le -\frac{c}{2\omega} + \frac{3\omega}{4}. \tag{5.5}$$

The equations in (i)–(iii) are derived in a standard way, see [9] for more details. The estimates (5.5) follow from the fact that, in the (c, R)-plane, the graph of $R(c)$ coincides with the concave separatrix trajectory of the system (5.4) contained in the triangle with vertices $(0,0)$, $(-\frac{1}{2}\omega^2, \omega)$ and $(0, \frac{3}{4}\omega)$ and connecting the first two of them. Note that $R'(c) = -1/(2\omega)$ at $c = -\frac{1}{2}\omega^2$. We also use analyticity of the ratio $R(c)$ at this point and properties of the phase portrait of (5.4) to verify the above statements.

After this preparation, we turn to prove the estimates we used in the preceding sections.

A2. The estimate for $A_2 = \frac{||r^2||^2}{||r||^2}$. By (4.2), (5.3) and the first inequality in (5.5), we have

$$A_2 = \frac{\int_0^T r^4 dt}{\int_0^T r^2 dt} = \frac{I_4'}{I_2'} = \frac{2cI_0' + 4\omega I_2'}{3I_2'} = \frac{2c}{3R} + \frac{4\omega}{3} \geq \omega.$$

B2. The estimate for $B_2 = \frac{||rr'||^2}{||r'||^2}$. As $I_6' = \dfrac{16\omega c}{15} I_0' + \left(\dfrac{6c}{5} + \dfrac{32\omega^2}{15}\right) I_2'$, we have by (3.4) as above

$$B_2 = \frac{\int_0^T r^2(c + \omega r^2 - \frac{1}{2}r^4)dt}{\int_0^T (c + \omega r^2 - \frac{1}{2}r^4)dt} = \frac{2cI_2' + 2\omega I_4' - I_6'}{2cI_0' + 2\omega I_2' - I_4'}$$

$$= \frac{2}{5}\frac{\omega c I_0' + (3c + 3\omega^2)I_2'}{2cI_0' + \omega I_2'} = \frac{2}{5}\frac{\omega c + (3c + 2\omega^2)R}{2c + \omega R} \geq \frac{4\omega}{5}.$$

To obtain the last inequality, we used both estimates in (5.5).

C2. The estimate for $C_2 = \lambda_2 + \lambda_0(1 + \frac{1}{d})\frac{||\psi||^2}{\nu^2||\psi_0||^2}$. By (3.12) and (3.13) we have

$$C_2 = \lambda_2 + \lambda_0 \left(1 + \frac{1}{d}\right)\left(\frac{||r||^2||\psi_0||^2}{\langle r, \psi_0\rangle^2} - 1\right).$$

Next,

$$\langle r, \psi_0\rangle = \int_0^T [3r^3 - (3\omega + \lambda_0)r]dt = 6I_3' - (6\omega + 2\lambda_0)I_1' = -2\lambda_0 I_1',$$

$$\begin{aligned} ||r||^2||\psi_0||^2 &= \textstyle\int_0^T r^2 dt \int_0^T (3r^2 - 3\omega - \lambda_0)^2 dt \\ &= 4I_2'[9I_4' - (18\omega + 6\lambda_0)I_2' + (3\omega + \lambda_0)^2 I_0'] \\ &= 4I_2'[(6c + (3\omega + \lambda_0)^2)I_0' - (6\omega + 6\lambda_0)I_2']. \end{aligned}$$

By (3.5), we have

$$I_2'(c) = \frac{1}{2}\int_0^T r^2 dt = (r_0^2/\alpha)\int_0^{K(k)} dn^2(t)dt = \sqrt{2}r_0 E(k). \tag{5.6}$$

Making use of the identity $E(k) = \frac{1}{2}\pi F(\frac{1}{2}, -\frac{1}{2}, 1, k^2)$ where F is the Gauss hypergeometric function, we obtain an appropriate expansion to estimate E from above

$$E(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3k^4}{64} - \frac{5k^6}{512} - \dots,\right), \quad E^2(k) \le \frac{\pi^2}{4}\left(1 - \frac{k^2}{2} - \frac{k^4}{32}\right)$$

with all removed terms negative. As $I_1' = \frac{1}{\sqrt{2}}\pi$, by (3.5) this implies

$$I_2'^2 \le -I_1'^2 \frac{\omega^2 - 10\omega r_0^2 + r_0^4}{8r_0^2}.$$

Together with $I_0' I_2' \ge I_1'^2$, this yields

$$\begin{aligned}\frac{||r||^2||\psi_0||^2}{\langle r, \psi_0\rangle^2} - 1 &\le \frac{1}{\lambda_0^2}\left[6c + (3\omega + \lambda_0)^2 + \frac{3}{4r_0^2}(\omega + \lambda_0)(\omega^2 - 10\omega r_0^2 + r_0^4)\right] - 1 \\ &= \frac{\lambda_2}{\lambda_0}\left(\frac{\lambda_2 + \omega}{4r_0^2} - 1\right) \le \frac{\lambda_2}{\lambda_0}\left(\frac{\sqrt{3}}{8} - 1\right)\end{aligned}$$

where the equality is obtained by direct calculations. Therefore,

$$C_2 \ge \lambda_2\left(-\frac{1}{d} + \frac{d+1}{d}\frac{\sqrt{3}}{8}\right) = \bar{C}_2\lambda_2$$

with $\bar{C}_2 > 0$ an absolute constant when $d \ge 4$ is fixed.

As a by-product of our calculations, we easily obtain also the estimate

$$\frac{\lambda_0}{\nu^2||\psi_0||^2} = \frac{\lambda_0||\psi_0||^2}{\langle r, \psi_0\rangle^2} \ge \frac{\lambda_2(\frac{\sqrt{3}}{8} - 1) + \lambda_0}{||r||^2} \ge -D_1\omega^{\frac{1}{2}}.$$

D2. The estimates for $D_{21} = \frac{||r^3 - \omega r||}{||r'||}$ and $D_{22} = \frac{||r'||}{||r||}$. We proceed as in case **D1** above. Making use of statements (i) and (ii) of the Lemma, we have

$$\begin{aligned}D_{21}^2 &= \frac{\int_0^T(\omega^2 r^2 - 2\omega r^4 + r^6)dt}{\int_0^T(c + \omega r^2 - \frac{1}{2}r^4)dt} = \frac{\omega^2 I_2' - 2\omega I_4' + I_6'}{cI_0' + \omega I_2' - \frac{1}{2}I_4'} \\ &= \frac{-4c\omega I_0' + (18c + 7\omega^2)I_2'}{5(2cI_0' + \omega I_2')} = \frac{-\omega I_0 + 3I_2}{I_0} \le 5\omega.\end{aligned}$$

Similarly,

$$\begin{aligned}D_{22}^2 &= \frac{\int_0^T(c + \omega r^2 - \frac{1}{2}r^4)dt}{\int_0^T r^2 dt} = \frac{cI_0' + \omega I_2' - \frac{1}{2}I_4'}{I_2'} \\ &= \frac{2cI_0' + \omega I_2'}{3I_2'} = \frac{(2c + \omega^2)I_0}{5I_2 - \omega I_0} \le \frac{2c + \omega^2}{3\omega}.\end{aligned}$$

As before, we used that $I_2(c)/I_0(c)$ is a decreasing function and calculated the value $I_2(0)/I_0(0) = \frac{4}{5}\omega$.

Acknowledgment. The first author has been partially supported by a Research grant DDVU 02/91 (2010) of the Bulgarian Ministry of Education and Science.

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