# ORBITAL STABILITY FOR PERIODIC STANDING WAVES OF THE KLEIN-GORDON-ZAKHAROV SYSTEM AND THE BEAM EQUATION

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Abstract. The existence and stability of spatially periodic waves $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ in the Klein-Gordon-Zakharov (KGZ) system are studied. We show a local existence result for low regularity initial data. Then, we construct a one-parameter family of periodic dnoidal waves for (KGZ) system when the period is bigger than $\sqrt{2}\pi$. We show that these waves are stable whenever an appropriate function satisfies the standard Grillakis-Shatah-Strauss type condition. We compute the intervals for the parameter $\omega$ explicitly in terms of $L$ and by taking the limit $L \to \infty$ we recover the previously known stability results for the solitary waves in the whole line case.

For the beam equation, we show the existence of spatially periodic standing waves and show that orbital stability holds if an appropriate functional satisfies Grillakis-Shatah-Strauss type condition.

## 1. Introduction and main results

In this paper we will be interested in the stability of standing wave solutions to certain partial differential equations.

### 1.1. The Klein-Gordon-Zakharov system.

We will consider first the Klein-Gordon-Zakharov system, which is given (in dimensionless parameters with $0 < c < 1$) as

$$(1) \qquad \begin{cases} u_{tt} - u_{xx} + u + uv = 0, & (t,x) \in \mathbf{R}^1 \times \mathbf{R}^1 \quad or \quad (t,x) \in \mathbf{R}^1 \times [-L, L] \\ v_{tt} - c^2 v_{xx} = (|u|^2)_{xx} \end{cases}$$

The Klein-Gordon-Zakharov system describes the interaction of a Langmuir wave and an ion sound wave in plasma. In our notations $u$ is the fast scale component of the electric field, whereas $v$ denotes the deviation of ion density, [18, 2]. Regarding the well-posedness theory, a lot of progress has been made in the last fifteen years. The first local well-posedness result seem to go back to Ginibre-Tsutsumi-Velo, [6] and Ozawa-Tsutaya-Tsutsumi, [16]. Since the solutions produced by [6] were constructed via a fixed point method in a Strichartz type space, they were only conditionally unique. In an interesting paper, Masmoudi and Nakanishi, [13] showed that under some extra smoothness assumptions, the solutions are also unconditionally unique (i.e. unique when considered in some large energy space). Interesting developments came about in the late nineties regarding global existence of solutions of (1). It turns out that the conservation laws associated with (1) (and its higher dimensional analogues) are good enough to only control small solutions, which were promptly shown to exist globally, [16], [17]. Note that here different propagation speeds and (high) dimension contributed to the success of these approaches.

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On the other hand, we have the following general theorem regarding local well-posedness of (1). Note that our interest is mainly in the periodic case, so we need a proper local well-posedness result for the periodic KGZ system, (1).

**Theorem 1.** *Let $\alpha > 1/2$. Then, the Cauchy problem for* (1)*,* **considered both for $x \in \mathbf{R}^1$ or in the periodic context** $0 < x < 1$ *is locally well-posed in the space $H^\alpha \times H^{\alpha-1} \times H^{\alpha-1} \times H^{\alpha-2}$.*

*More precisely, given $(u(0), u_t(0)) \in H^\alpha \times H^{\alpha-1}$, $(v(0), v_t(0)) \in H^{\alpha-1} \times H^{\alpha-2}$, there exists a time $T$, depending only on the norms in the respective spaces, so that there exists an unique solution $u(t) \in C([0,T], H^\alpha), u_t \in C([0,T], H^{\alpha-1}), v(t) \in C([0,T], H^{\alpha-1}), v_t \in C([0,T], H^{\alpha-2})$.*

*Moreover, the solution mapping $S(t)(u(0), u_t(0), v(0), v_t(0)) = (u(t), v(t))$ is Lipschitz in the respective norms.*

Next, we discuss standing wave solutions to (1). We will fix in what follows $c = 1$ in the second equation for simplicity, noting that the case when $c \neq 1$ can be treated similarly. As we have already mentioned, there have been lots of results in this direction, mostly for the higher dimensional case. To be more precise, we are considering solutions of (1) in the form

$$u(t,x) = e^{iwt}\varphi_w(x), \quad v(t,x) = \psi(x), \tag{2}$$

where $\phi(x)$ and $\psi(x)$ are either real-valued periodic functions with fixed fundamental period $L$ or vanishing at infinity functions (in the whole line context). Substituting (2) in (1) leads to the system

$$\begin{cases} -w^2\varphi'_w(x) - \varphi''_w(x) + \varphi_w(x) + \varphi_w(x)\psi(x) = 0 \\ \\ -\psi''(x) = (\varphi_w^2)_{xx}. \end{cases} \tag{3}$$

In the whole line scenario, this implies $\psi = -\varphi_\omega^2$ and consequently, the first equation becomes the standard second order ODE

$$-\varphi'' + (1 - w^2)\varphi - \varphi^3 = 0,$$

which is known to have unique (up to translation) *sech* solution, whenever $w \in (-1, 1)$. In fact, these are explicitly found[1] in the work of Chen, [1] as follows

$$\begin{aligned} \varphi_w(x) &= \sqrt{2(1-w^2)}sech(x\sqrt{1-w^2}) \\ \psi_w(x) &= -2(1-w^2)sech^2(x\sqrt{1-w^2}) \end{aligned} \tag{4}$$

Similar results hold in higher dimensions, that is one can produce an unique radial and radially decreasing function $\varphi_w$ (for which there is unfortunately no explicit formulas available, when $n \geq 2$), so that $(e^{iwt}\varphi_w, -\varphi_w^2)$ is a solution of (1), whenever $w \in (-1, 1)$. For these particular solutions, Gan, [3], Gan-Zhang, [5] and subsequently Ohta-Todorova, [15] have shown very strong instability results. Namely, in dimensions $n = 2, 3$ and for $c \neq 1$, $w \in (-1, 2)$, there are solutions that start very close to the standing wave $(e^{iwt}\varphi_w, -\varphi_w^2)$, which either blow up in finite time or else $\lim_{t\to\infty} \|(u(t), v(t))\| = \infty$. These results should of course be contrasted with the "global regularity for small data" results in [16, 17], that we have alluded to earlier.

Our interest is in the orbital stability of standing waves in the one-dimensional periodic case. Before we continue with the existence results (which are slightly more delicate in the periodic case, due to the fact that there is one more integration constant in the second equation), let us give the following

[1] for the case $c = 1$, the general case could be treated in a similar way

**Definition 1.** *A standing wave solution for (1), of the form* $(e^{iwt}\varphi(x), \psi(x))$*, is said to be orbitally stable in* $H^1(\mathbf{R}^1) \times L^2(\mathbf{R}^1)$*, if for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $(u_0, \rho_0, v_0, n_0) \in X = H^1 \times L^2 \times L^2 \times L^2$ *satisfies* $||(u_0, \rho_0, v_0, n_0) - (\varphi, iw\varphi, \psi, 0)||_X < \delta$*, then the solution to* (1) *with initial data* $u(x,0) = u_0, u_t(x,0) = \rho_0, v(0,x) = \psi, v_t(x,0) = n_0$ *satisfies*

$$\sup_{t>0} \inf_{\theta\in[0,2\pi];y\in\mathbf{R}^1} \|(u,v) - (e^{i\theta}e^{iwt}\varphi(\cdot+y), \psi(\cdot+y))\|_{H^1(\mathbf{R}^1)\times L^2(\mathbf{R}^1)} < \varepsilon.$$

The question for orbital stability of the waves described in (4) was addressed by Chen, [1]. He proved orbital stability for $(e^{iwt}\varphi_w, \psi_w)$, provided $1 > |w| > \frac{\sqrt{2}}{2}$.

We collect our existence results in the following

**Proposition 1.** *Let* $L > \sqrt{2}\pi$ *be fixed. Then, for every* $\omega \in (0,1)$*, there is a smooth dnoidal periodic standing wave solution of* (3)*,* $(e^{iwt}\varphi_\omega, \psi_\omega) \in H^\infty[0,L] \times H^\infty[0,L]$*, where* $\varphi_\omega$ *is described in* (20) *and* (21) *and* $\psi_\omega = -\varphi_\omega^2$*.*

The next result, which is the main result of this work, describes the orbital stability of the spatially periodic standing waves in Proposition 1.

**Theorem 2.** *Let* $L > \sqrt{2}\pi$ *be a given period. Then, there is orbital stability for all* $\omega$ *satisfying the inequality*

$$\sqrt{-\frac{G(\kappa_0(L))}{F(\kappa_0(L))}} \leq |\omega| \leq \sqrt{1 - \frac{2\pi^2}{L^2}} \tag{5}$$

*where*

$$F(\kappa) = \left[2(2-\kappa^2)E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)(1-\kappa^2)K^2(\kappa)\right]$$
$$G(\kappa) = 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)E^2(\kappa)$$

*where* $E(\kappa), K(\kappa)$ *are the standard elliptic functions (see Section 3 for definitions and notations),* $\kappa_0(L)$ *is the inverse function to the increasing function*

$$\kappa \to \frac{2\sqrt{2-\kappa^2}K(\kappa)}{\sqrt{1+\frac{G(\kappa)}{F(\kappa)}}}, \quad \kappa \in (0,1)$$

**Remarks:**

(1) We establish in Section 3 below, that for $L < \sqrt{5}\pi$, the solution set of the inequalities (5) is empty. More precisely, the inverse function $k_0(L)$ is defined only in $(\sqrt{5}\pi, \infty)$, because the range of $\kappa \to \frac{2\sqrt{2-\kappa^2}K(\kappa)}{\sqrt{1+\frac{G(\kappa)}{F(\kappa)}}}$ is $(\sqrt{5}\pi, \infty)$.
(2) For every $L > \sqrt{5}\pi$, the inequality (5) has a whole interval of solutions.
(3) One can obtain Chen's result, [1] for orbital stability of the waves (4). Namely, since

$$\lim_{L\to\infty} \kappa_0(L) = 1, \lim_{L\to\infty} \sqrt{-\frac{G(\kappa_0(L))}{F(\kappa_0(L))}} = \frac{1}{\sqrt{2}},$$

which combined with (5) yields the range $|w| > \frac{1}{\sqrt{2}}$. We discuss the details in Section 3.

1.2. **The beam equation.** In this paper we would like to discuss also the standing wave solutions of the so-called beam equation,

$$u_{tt} + \Delta^2 u + u - |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^d \text{ or } (t,x) \in \mathbf{R}^1 \times [-L,L]^d, \tag{6}$$

where $p > 1, L > 0$ and we either require periodic boundary conditions (in the case $x \in [-L,L]$) or vanishing at infinity for $x \in \mathbf{R}^d$. This equation goes back to a work of McKenna and Walter, [14], where it was proposed as a model for suspension bridges. One can still consider the standing wave ansatz $u(t,x) = e^{i\omega t}\varphi_\omega(x)$ in (6), whence we get the following ODE for the real function $\varphi$

$$\Delta^2\varphi + (1-\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0. \tag{7}$$

Note that we do not expect positivity of the solution $\varphi$, due to the fact that the bilaplacian $\Delta^2$ does not obey the maximum principle. Smooth and rapidly decaying solutions to (7) were shown to exists in the whole line case by Levandosky, [11], provided $\omega \in (-1,1)$. In fact, it is easy to see by scaling arguments that they exhibit the following dependence on the parameter $\omega$, $\varphi_\omega = (1-\omega^2)^{\frac{1}{p-1}}\varphi_0((1-\omega^2)^{\frac{1}{4}}x)$.

Based on that, Levandosky has concluded (see Section 7, [11]), using the Grillakis-Shatah-Strauss theory, that these waves are orbitally unstable for $p \geq 9$, while for $p < 9$, there is orbital stability for $1 > |\omega| > \sqrt{\frac{2(p-1)}{p+7}}$ and orbital instability for $0 \leq |\omega| \leq \sqrt{\frac{2(p-1)}{p+7}}$.

In this paper, we will consider the orbital stability of spatially periodic standing waves of (6). Our first result concerns the existence of such waves.

**Proposition 2.** *Let $\omega \in (-1,1)$, $L > 0$ and $1 < p < \frac{2d}{d-4} - 1$, if $p \geq 5$. Then, the equation (7), considered in $[-L,L]^d$, with periodic boundary conditions has a smooth solution $\varphi$.*

Next, we are interested in a criteria for stability of the solutions produced in Proposition 2. We need to introduce a few notations, before we can state the main result, which ties the stability of such waves to the convexity of a function $d(\omega)$.

Let $v = u_t$ and $\mathbf{u} = (u,v)$. Introduce the functionals

$$\begin{aligned} E(\mathbf{u}) &= \int (\frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|u|^2 - \frac{1}{p-1}|u|^{p+1})dx \\ Q(\mathbf{u}) &= Im\int \overline{u}v dx \end{aligned}$$

Define the function

$$d(\omega) = E(\varphi) - \omega Q(\varphi). \tag{8}$$

We have the following

**Theorem 3.** *Let $\omega, L, p$ satisfy the requirements of Proposition 2. Then, the solutions constructed there are orbitally stable, if $d''(\omega) > 0$.*

## 2. Local well-posedness for the KGZ system: Proof of Theorem 1

In this section, we give some standard preliminary results, after which, we present the Proof of Theorem 1.

2.1. **Preliminaries.** The Fourier transform of a smooth and decaying function on $\mathbf{R}^1$ and its inverse are given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx$$
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi.$$

For the periodic case, for $f \in L^2[0,1]$, the corresponding formulas are given by

$$a_n = \int_0^1 f(x) e^{-2\pi i n x} dx$$
$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

It is convenient to introduce[2] the following operators for $s \in \mathbf{R}^1$, namely for Schwartz functions on $\mathbf{R}^1$, $\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi)$ and $\widehat{\langle\nabla\rangle^s f}(\xi) := (1+|\xi|^2)^{s/2}\hat{f}(\xi)$. For the periodic case, we take $\langle\nabla\rangle^s f(x) = \sum_n a_n \langle n\rangle^s e^{2\pi i n x}$.

Next, let $\Upsilon$ be a smooth and even function, so that $\Upsilon(\xi) = 1, |\xi| < 1/2, \Upsilon(\xi) = 0, |\xi| > 1$. Let $\chi(\xi) := \Upsilon(\xi/2) - \Upsilon(\xi)$, so that $supp \chi \subset \{1/2 < |\xi| < 2\}$ and $\Upsilon(\xi) + \sum_{k=1}^{\infty} \chi(2^{-k}\xi) = 1$ for all $\xi \neq 0$.

This of course is a partition of unity, whence we could define the Littlewood-Paley "projections" $\widehat{P_k f}(\xi) = \chi(2^{-k}\xi)\hat{f}(\xi)$, $\widehat{P_{\leq k} f}(\xi) = \Upsilon(2^{-k}\xi)\hat{f}(\xi)$ and $P_{\leq 0} + \sum_{k=1}^{\infty} P_k = Id$.

For the periodic case, we take

$$P_k f(x) := \sum_n a_n \chi(2^{-k} n) e^{2\pi i n x}$$
$$P_{\leq k} f(x) := \sum_{n \leq 2^k} a_n \Upsilon(2^{-k} n) e^{2\pi i n x}$$

In both the periodic and non-periodic cases, it is easy to see that
$f_k := P_k f(x) = \int_{-\infty}^{\infty} \hat{\chi}(\xi) f(x + 2^{-k}\xi) d\xi$, whence

$$\|P_k\|_{L^p \to L^p} \leq \|\hat{\chi}\|_{L^1}, \|P_{\leq k}\|_{L^p \to L^p} \leq \|\hat{\Upsilon}\|_{L^1} \quad 1 \leq p \leq \infty.$$

A basic property that is an immediate corollary of the boundedness of $P_k, P_{\leq k}$ is
$\||\nabla|^s P_k f\|_{L^p} \sim 2^{ks}\|P_k f\|_{L^p}$, $\||\nabla|^s P_{\leq k} f\|_{L^p} \lesssim 2^{ks}\|P_{\leq k} f\|_{L^p}$ $1 \leq p \leq \infty$.

The following fundamental property of the Littlewood-Paley operators will be useful in the sequel. Namely, letting $l \geq 3$, we have

$$P_{\leq l-3}[f g_l] = P_{\leq l-3}[f_{l-2 \leq \cdot \leq l+2} g_l] \tag{9}$$

We now define the Sobolev spaces $W^{s,r}$, $s \in \mathbf{R}^1, 1 < r < \infty$, via the norm $\|f\|_{W^{s,r}} := \|\langle\nabla\rangle^s f\|_{L^r}$. Using Littlewood-Paley operators, one can write an equivalent norm in the form

$$\|f\|_{W^{s,r}} \sim \|f_{\leq 0}\|_{L^r} + \left\| \left( \sum_{k=1}^{\infty} 2^{2ks} |f_k|^2 \right)^{1/2} \right\|_{L^r}, \quad 1 < r < \infty.$$

[2] whenever it makes sense

The particular case $r = 2$ is important and it is denoted by $H^s := W^{s,2}$. We are now ready to state the Sobolev embedding theorem and the related Bernstein inequalities. Namely, for $1 < p < q < \infty$ , we have (both in the periodic and non-periodic case)

$$\|f\|_{L^q} \leq C_{p,q} \|f\|_{W^{\frac{1}{p}-\frac{1}{q},p}}. \tag{10}$$

In the special cases, when $q = \infty$, (10) of course fails, but we still have $\|f\|_{L^\infty} \leq C_{p,s} \|f\|_{W^{s,p}}$, whenever $s > \frac{1}{p}$. Finally, when we are dealing with Littlewood-Paley localized functions and for all $1 \leq p < q \leq \infty$ , we have the Bernstein inequality,

$$\|f_k\|_{L^q} + \|f_{\leq k}\|_{L^q} \lesssim 2^{k(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}. \tag{11}$$

We will often use the following Besov space $B^s_{r,2}$, whose norm may be introduced as

$$\|f\|_{B^s_{r,2}} = \|f_{\leq 0}\|_{L^r} + \left( \sum_{k=1}^\infty 2^{2ks} \|f_k\|^2_{L^r} \right)^{1/2},$$

Note that by the triangle inequality, for $r \geq 2$, $\|f\|_{W^{s,r}} \leq C\|f\|_{B^s_{r,2}}$. We also need to use the mixed Lebesgue spaces $L^q_t W^{s,r}_x$, which are defined via the norm $\|f\|_{L^q_t W^{s,r}_x} := \|\|f\|_{W^{s,r}_x}\|_{L^q_t}$.

2.2. **Energy estimates.** We have the following standard energy estimates

**Lemma 1.** *Let $\psi$ satisfy the linear inhomogeneous Klein-Gordon equation, while $\phi$ satisfies the wave equation, i.e.*

$$\psi_{tt} - \psi_{xx} + \psi = F$$
$$\phi_{tt} - \phi_{xx} = G$$

*where $x \in \mathbf{R}^1$ or $x \in [0, 1]$. For any $\alpha \in \mathbf{R}^1$, we have the following estimates*

$$\|\psi(t)\|_{L^\infty_t[0,T]H^\alpha} + \|\partial_t \psi(t)\|_{L^\infty_t[0,T]H^{\alpha-1}} \lesssim \|\psi(0)\|_{H^\alpha} + \|\psi_t(0)\|_{H^{\alpha-1}} + \|F\|_{L^1_t[0,T]H^{\alpha-1}_x} \tag{12}$$

$$\|\phi(t)\|_{L^\infty_t[0,T]H^\alpha} + \|\partial_t \phi(t)\|_{L^\infty_t[0,T]H^{\alpha-1}} \lesssim \|\phi(0)\|_{H^\alpha_x} + \|\phi_t(0)\|_{H^{\alpha-1}} + \||\nabla|^{-1} G\|_{L^1_t[0,T]H^\alpha_x} \tag{13}$$

2.3. **Proof of Theorem 1.** We now study the local well-posedness issue for (1). Our method will consists of showing the existence of a fixed point argument in the space $(u, u_t; v, v_t) \in X_T \times Y_T$, where $X = C([0, T], H^\alpha \times H^{\alpha-1})$ and $Y = C([0, T], H^{\alpha-1} \times H^{\alpha-2})$. More precisely, for (1), we consider left-hand sides of the form $F(u, v) = -uv$, $G(u, v) = G(u) = \partial_{xx}(|u|^2)$. We thus have by (12) and (13)

$$\|u\|_X \lesssim \|u(0), u_t(0)\|_{H^\alpha \times H^{\alpha-1}} + \|F(u, v)\|_{L^1_T H^{\alpha-1}}$$
$$\|v\|_Y \lesssim \|v(0), v_t(0)\|_{H^{\alpha-1} \times H^{\alpha-2}} + \||\nabla|^{-1} G(u)\|_{L^1_T H^{\alpha-1}}$$

Note that since $\|F\|_{H^{\alpha-1}} = \|uv\|_{H^{\alpha-1}}$ and $\||\nabla|^{-1} G(u)\|_{H^{\alpha-1}} \leq C\|u\bar{u}\|_{H^\alpha}$, the fixed point argument will give a time $T > 0$ and the existence of an unique (in $X \times Y$) solution $(u, v)$, provided we can verify the following estimates for $\alpha > 1/2$,

$$\|uv\|_{H^{\alpha-1}} \leq C\|u\|_{H^\alpha} \|v\|_{H^{\alpha-1}} \tag{14}$$

$$\|uv\|_{H^\alpha} \leq C\|u\|_{H^\alpha} \|v\|_{H^\alpha} \tag{15}$$

Note that we need to establish these two estimates both in the periodic and non-periodic context.

The inequality (15) is equivalent to the well-known fact that $H^\alpha, \alpha > \frac{1}{2}$ is a Banach algebra, both in the periodic and non-periodic context. Thus, we concentrate on the proof of (14), which is not difficult either. In fact, a version of it, on the circle appears as Lemma 4, [9], which is why we will only pursue its proof on the line.

By splitting in high and low frequencies, it will suffice to show that for all $k \geq 1$ and $\alpha > 1/2$,

$$\|(uv)_{\leq 0}\|_{L^2} \leq C\|u\|_{H^\alpha}\|v\|_{H^{\alpha-1}} \tag{16}$$

$$\sum_{k=1}^{\infty} 2^{2(\alpha-1)k}\|P_k[uv]\|_{L^2}^2 \leq C\|u\|_{H^\alpha}^2\|v\|_{H^{\alpha-1}}^2. \tag{17}$$

2.3.1. *Proof of* (16). Write $P_{\leq 0}[uv] = P_{\leq 0}[uv_{\leq 3}] + P_{\leq 0}[uv_{>3}]$. For the first term, the estimate is rather direct, since[3])

$$\|P_{\leq 0}[uv_{\leq 3}]\|_{L^2} \leq C\|uv_{\leq 3}\|_{L^1} \leq C\|u\|_{L^2}\|v_{\leq 3}\|_{L^2} \leq C\|u\|_{H^\alpha}\|v\|_{H^{\alpha-1}}.$$

Regarding the other term under consideration, we have

$$P_{\leq 0}[uv_{>3}] = \sum_{l=4}^{\infty} P_{\leq 0}[uv_l] = \sum_{l=4}^{\infty} P_{\leq 0}[u_{l-2\leq l+2} \cdot v_l]$$

Thus,

$$\|P_{\leq 0}[uv_{>3}]\|_{L^2} \leq \sum_{l=4}^{\infty} \|P_{\leq 0}[u_{l-2\leq l+2} \cdot v_l]\|_{L^2} \leq C\sum_{l=4}^{\infty} \|u_{l-2\leq l+2} \cdot v_l\|_{L^1} \leq$$
$$\leq C(\sum_{l=4}^{\infty} 2^{2\alpha l}\|u_{l-2\leq l+2}\|_{L^2}^2)^{1/2}(\sum_{l=4}^{\infty} 2^{-2\alpha l}\|v_l\|_{L^2}^2)^{1/2} \leq C\|u\|_{H^\alpha}\|v\|_{H^{\alpha-1}},$$

where we have applied the Bernstein's inequality (11) (with $k = 0$, $q = 2$, $p = 1$), Hölder's inequality and the fact that $-\alpha < \alpha - 1$ (whence $(\sum_{l=4}^{\infty} 2^{-2\alpha l}\|v_l\|_{L^2}^2)^{1/2} \leq (\sum_{l=4}^{\infty} 2^{2(\alpha-1)l}\|v_l\|_{L^2}^2)^{1/2} \leq C\|v\|_{H^{\alpha-1}}$).

2.3.2. *Proof of* (17). Write

$$P_k[uv] = P_k[u\ v_{<k-3}] + P_k[u\ v_{k-3\leq\cdot\leq k+3}] + P_k[u\ v_{>k+3}]$$

The middle term is easy to handle by

$$\sum_{k=1}^{\infty} 2^{2(\alpha-1)k}\|P_k[u\ v_{k-3\leq\cdot\leq k+3}]\|_{L^2}^2 \leq \sum_{k=1}^{\infty} 2^{2(\alpha-1)k}\|v_{k-3\leq\cdot\leq k+3}\|_{L^2}^2\|u\|_{L^\infty}^2 \leq C\|v\|_{H^{\alpha-1}}^2\|u\|_{H^\alpha}^2,$$

where we have used the Sobolev embedding estimate $\|u\|_{L^\infty} \leq C\|u\|_{H^\alpha}$.

The first term (or the high-low interaction term) can be dealt with as follows. Observe first that $P_k[u\ v_{<k-3}] = P_k[u_{k-1\leq\cdot\leq k+1}\ v_{<k-3}]$. Thus an application of (11) again yields

$$\sum_{k=1}^{\infty} 2^{2(\alpha-1)k}\|u_{k-1\leq\cdot\leq k+1}\ v_{<k-3}\|_{L^2}^2 \leq C\sum_{k=1}^{\infty} 2^{2\alpha k}\|u_{\sim k}\|_{L^2}^2 \sup_{k\geq 1} 2^{-2k}\|v_{<k-3}\|_{L^\infty}^2$$
$$\leq C\|u\|_{H^\alpha}^2 \sup_{k\geq 1} 2^{-3k/2}\|v_{<k-3}\|_{L^2}^2.$$

We only need to observe that since $-\frac{3}{4} < \alpha - 1$, we have

$$\sup_{k\geq 1} 2^{-3k/4}\|v_{<k-3}\|_{L^2} \lesssim \|v_{\leq 0}\|_{L^2} + \sup_{m\geq 1} 2^{(\alpha-1)m}\|v_m\|_{L^2} \leq C\|v\|_{H^{\alpha-1}}$$

[3]In fact, this last estimate holds up for any $\alpha \geq 0$.

Finally, the high-high term is handled as follows. Note that $P_k[u\ v_{>k+3}] = \sum_{l=k+4}^{\infty} P_k[u_{l-1\leq\cdot\leq l+1} v_l]$. In view of the embedding $l^1 \hookrightarrow l^2$ and the Bernstein estimate (11), we have

$$(\sum_{k=1}^{\infty} 2^{2(\alpha-1)k} \|P_k[u\ v_{>k+3}]\|_{L^2}^2)^{1/2} \leq \sum_{k=1}^{\infty} 2^{(\alpha-1)k} \|P_k[u\ v_{>k+3}]\|_{L^2} \leq$$
$$\leq C \sum_{k=1}^{\infty} \sum_{l=k+4}^{\infty} 2^{(\alpha-\frac{1}{2})k} \|u_{l-1\leq\cdot\leq l+1} v_l\|_{L^1}.$$

Interchanging the order of the $l$ and $k$ summation and taking into account $\alpha > \frac{1}{2}$ yields the bound $\sum_{l=5}^{\infty} 2^{(\alpha-\frac{1}{2})l} \|u_{l-1\leq\cdot\leq l+1}\|_{L^2} \|v_l\|_{L^2}$. Thus, we continue the estimation by

$$\sum_{l=5}^{\infty} 2^{(\alpha-\frac{1}{2})l} \|u_{l-1\leq\cdot\leq l+1}\|_{L^2} \|v_l\|_{L^2} \leq (\sum_{l=5}^{\infty} 2^{2\alpha l} \|u_{l-1\leq\cdot\leq l+1}\|_{L^2}^2)^{1/2} (\sum_{l=5}^{\infty} 2^{-l} \|v_l\|_{L^2}^2)^{1/2}.$$

It now remains to observe that since $-l < 2(\alpha-1)l$ for $l \geq 1$ and $\alpha > \frac{1}{2}$, we have $(\sum_{l=5}^{\infty} 2^{-l} \|v_l\|_{L^2}^2)^{1/2} \leq C\|v\|_{H^{\alpha-1}}$. The proof of Theorem 1 is complete.

## 3. Orbital stability for the standing waves of the Klein-Gordon-Zakharov system

In this section, we outline the existence results of Proposition 1, after which we present the proof of Theorem 2.

3.1. **Proof of Proposition 1.** Integrating twice the second equation in (3) and taking the constant of integrations to be zero, we get $\psi(x) = -\varphi^2(x)$. Then, $\varphi(x)$ must satisfy the equation

$$-\varphi''(x) + c\varphi(x) - \varphi^3(x) = 0, \tag{18}$$

where $c = 1 - w^2 > 0$. Integrating this equation, we obtain

$$\varphi'^2 = a + 2c\varphi - \varphi^4. \tag{19}$$

Hence, the periodic solution is given by the periodic trajectories $H(\varphi, \varphi') = a$ of the Hamiltonian vector field $dH = 0$, where

$$H(x, y) = y^2 + x^4 - 2cx^2.$$

It is well-known that the equation (18) has dnoidal wave solutions given by

$$\varphi(x) = \varphi(x, \eta_1, \eta_2)) = \eta_1 dn\left(\frac{\eta_1}{\sqrt{2}} x; \kappa\right), \tag{20}$$

where $\eta_1 > \eta_2 > 0$ are the positive zeros of the polynomial $-t^4 + 2ct^2 + a$ and

$$\kappa^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \ \ \eta_1^2 + \eta_2^2 = 2c. \tag{21}$$

Since the elliptic function $dn$ has fundamental period $2K$, where $K = K(\kappa)$ is the complete elliptic integral of the first kind, the function $\varphi$ given in (20) has fundamental period

$$L = L_\varphi = \frac{2\sqrt{2}}{\eta_1} K(\kappa). \tag{22}$$

By (21), one also obtains $c = \frac{\eta_1^2(2-\kappa^2)}{2}$ and

$$L = \frac{2\sqrt{2-\kappa^2}K(\kappa)}{\sqrt{c}}, \quad \kappa \in (0,1), \quad L \in I = \left(\frac{\sqrt{2}\pi}{\sqrt{c}}, \infty\right). \tag{23}$$

**Lemma 2.** *For any $c > 0$ and $L \in I$, there is a constant $a = a(c)$ such that the periodic solution (20) determined by $H(\varphi, \varphi') = a(c)$ has period $L$. The function $a(c)$ is differentiable.*

Proof. It is easily seen that the period $L$ is a strictly increasing function of $\kappa$:

$$\frac{d}{d\kappa}(\sqrt{2-\kappa^2}K(\kappa)) = \frac{(2-\kappa^2)K'(\kappa) - \kappa K(\kappa)}{\sqrt{2-\kappa^2}} = \frac{K'(\kappa) + E'(\kappa)}{\sqrt{2-\kappa^2}} > 0.$$

Moreover,

$$\frac{\partial L}{\partial a} = \frac{dL}{d\kappa}\frac{d\kappa}{da} = \frac{1}{2\kappa}\frac{dL}{d\kappa}\frac{d\kappa^2}{da}.$$

Further, we have

$$\frac{d(\kappa^2)}{da} = \frac{d(\kappa^2)}{d(\eta_1^2)}\frac{d(\eta_1^2)}{da} = \frac{c}{\eta_1^4(\eta_1^2 - c)}.$$

We see that $\partial L(a,c)/\partial a \neq 0$, therefore the implicit function theorem yields the result.

3.2. **Proof of Theorem 2: Preliminaries.** Rewrite (1) as

$$\begin{cases} u_t = -\rho \\ \rho_t = -u_{xx} + u + vu \\ v_t = n_x \\ n_t = v_x + (|u|^2)_x. \end{cases} \tag{24}$$

System (24) can be written as a Hamiltonian system of the form

$$\frac{d}{dt}U(t) = JE'(t), \tag{25}$$

where $U = (u, \rho, n, v)$, $J$ is the skew-symmetric linear operator

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\partial_x \\ 0 & 0 & 2\partial_x & 0 \end{pmatrix}$$

and $E$ is the energy functional given by

$$E(U) = \frac{1}{2}\int_0^L \left(|u_x|^2 + |u|^2 + v|u|^2 + |\rho|^2 + \frac{1}{2}v^2 + \frac{1}{2}n^2\right)dx.$$

Note, that the system (24) is invariant under the one-parameter group of unitary operator defined by $T(s)\vec{u} = (e^{-is}u, e^{-is}\rho, v, n)$ and the functional

$$Q(U) = Im\int_0^L (\rho\bar{u})dx$$

is a conserved quantity of the system (24). Denote by $\Phi_w = (\varphi, 0, 0, -w\varphi, \psi, 0)$, where $\varphi$ is the dnoidal wave given by (20). By direct computation, we see that $\Phi_w$ is a critical point of the functional $E + wQ$, that is

$$E'(\Phi_w) + wQ'(\Phi_w) = 0. \tag{26}$$

Define an operator

$$H_w(\Phi_w) = E''(\Phi_w) + wQ''(\Phi). \tag{27}$$

By direct computation, we have

(28)

$$\langle H_w \vec{u}, \vec{u} \rangle$$

$$= \left\langle \tilde{L_1} \begin{pmatrix} u_1 \\ u_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_4 \end{pmatrix} \right\rangle + \left\langle \tilde{L_2} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \right\rangle + \tfrac{1}{2}\int_0^L (2\varphi u_1 + u_5)^2 dx + \tfrac{1}{2}\int_0^L u_6^2 dx,$$

where $\tilde{L_1} = \begin{pmatrix} -\partial_x^2 - 3\varphi^2 + 1 & w \\ w & 1 \end{pmatrix}$ and $\tilde{L_2} = \begin{pmatrix} -\partial_x^2 - \varphi^2 + 1 & -w \\ -w & 1 \end{pmatrix}$.

**Lemma 3.** *(1) The operator $L_1 = -\partial_x^2 - 3\varphi^2 + c$ defined in $L^2_{per}[0, L]$ with domain $H^2_{per}[0, L]$ has exactly one negative eigenvalue, which is simple. In addition, zero is a simple eigenvalue.*

*(2)The operator $L_2 = -\partial_x^2 - \varphi^2 + c$ defined in $L^2_{per}[0, L]$ with domain $H^2_{per}[0, L]$ has no negative eigenvalues and zero is an eigenvalue which is simple.*

**Lemma 4.** *(1) The first three eigenvalues of the operator $\tilde{L_1}$ are simple and zero is the second eigenvalue.*

*(2) The operator $\tilde{L_2}$ has no negative eigenvalues, zero is the first eigenvalue and it is simple.*

To prove (1), consider the quadratic form

$$V_1(\vec{f}, \vec{f}) = \left\langle \tilde{L_1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle = \langle L_1 f_1, f_1 \rangle + \int_0^L (w f_1 + f_2)^2 dx.$$

From Lemma 3, there exists $\lambda_0 < 0$ and $f_0 \in H^2_{per}[0, L]$ satisfying $L_1 f_0 = \lambda_0 f_0$. Thus by choosing $f_1 = f_0$ and $f_2 = -w f_0$, we get that the first eigenvalue $\tilde{\lambda_0}$ of $\tilde{L_1}$ is negative. If $\tilde{\lambda_1}$ denotes the second eigenvalue of $\tilde{L_1}$, then by min-max characterization of eigenvalues, we have

$$\tilde{\lambda_1} = \max_{f_1, f_2} \min_{h_1 \perp f_1, h_2 \perp f_2} \frac{V_1(\vec{f}, \vec{f})}{||\vec{f}||^2}.$$

Taking $f_1 = f_0$ and $f_2 = -w f_0$, we get

$$\tilde{\lambda_1} \geq \min_{h_1 \perp f_1, h_2 \perp f_2} \frac{V_1(\vec{f}, \vec{f})}{||\vec{f}||^2}.$$

From Lemma 3, zero is the second eigenvalue of $L_1$, and the above inequality leads that $\tilde{\lambda_1} = 0$. Again using min-max characterization of eigenvalues and Lemma 3, we obtain that the third eigenvalue of $\tilde{L_1}$ is strictly positive.

To show (2), consider the quadratic form

$$V_2(\vec{g},\vec{g}) = \langle \tilde{L_2}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\rangle = \langle L_2 g_1, g_1\rangle + \int_0^L (-wg_1+g_2)^2 dx.$$

From Lemma 3 the operator $L_2$ has no negative eigenvalues and zero is the first eigenvalue. The proof follows from min-max characterization of eigenvalues as above. This finishes off the proof of the Lemma.

From Lemma 4 we obtain the following:

(1) The operator $H_w$ has exactly one negative eigenvalue, and $\mathcal{N}$-the negative eigenspace of $H_w$, is one-dimensional.

(2) For $\vec{f} = (\varphi', 0, 0, -w\varphi', 0, 0)$ and $\vec{g} = (0, \varphi, w\varphi, 0, 0, 0)$, the set $\mathcal{Z} = \{\alpha\vec{f} + \beta\vec{g}\}$ is the kernel of the operator $H_w$.

(3) There exists a closed subspace $\mathcal{P}$, such that $\langle H_w\vec{u}, \vec{u}\rangle \geq \delta_0||\vec{u}||$, for all $\vec{u} \in \mathcal{P}$.

Therefore, from (1)-(3), we obtain the following orthogonal decomposition of the $X$

$$X = \mathcal{N} \oplus \mathcal{Z} \oplus \mathcal{P}. \tag{29}$$

3.3. **Proof of Theorem 2: Conclusion.** We have that $\mathcal{N}$ is one-dimensional. The proof of theorem follows from the abstract stability theorem of Grillakis, Shatah and Strauss, provided we are able to show that $d''(w) > 0$, where $d(w) = E(\Phi_w) + wQ(\Phi_w)$. From (26) and using that

$$\int_0^L \varphi^2 dx = \sqrt{2}\eta_1 \int_0^{2K(\kappa)} dn^2(x;\kappa)dx = \frac{8K(\kappa)}{L}\int_0^{K(\kappa)} dn^2(x;\kappa)dx = \frac{8}{L}K(\kappa)E(\kappa),$$

we have

$$d'(w) = Q(\Phi_w) = -\int_0^L w\varphi^2 dx$$

and

$$\begin{aligned} d''(w) &= -\tfrac{8}{L}K(\kappa)E(\kappa) - \tfrac{8w}{L}\tfrac{d}{d\kappa}(K(\kappa)E(\kappa))\tfrac{d\kappa}{dc}\tfrac{dc}{dw} \\ &= \tfrac{8}{L}\left(-K(\kappa)E(\kappa) + 2w^2\tfrac{d}{d\kappa}(K(\kappa)E(\kappa))\tfrac{d\kappa}{dc}\right), \end{aligned} \tag{30}$$

Differentiating (21) and (22) with respect to $c$, we get

$$\frac{d\kappa}{dc} = \frac{1}{2\kappa}\frac{2\eta_2^2 - 4c\eta_2\eta_2'}{(2c-\eta_2^2)^2}, \tag{31}$$

$$\eta_2\eta_2' = \frac{K(\kappa) - \frac{d\kappa}{dc}K'(\kappa)(2c-\eta_2^2)}{K(\kappa)}. \tag{32}$$

From (31) and (32), we obtain

$$\frac{d\kappa}{dc} = \frac{K(\kappa)}{2cK'(\kappa) - \kappa(2c-\eta_2^2)K(\kappa)} \tag{33}$$

Finally for $d''(w)$ using that $\eta_2^2 = \frac{2c(1-\kappa^2)}{2-\kappa^2}$ and $K'(\kappa) = \frac{E(\kappa)-(1-\kappa^2)K(\kappa)}{\kappa(1-\kappa^2)}$, $E'(\kappa) = \frac{E(\kappa)-K(\kappa)}{\kappa}$, we obtain

$$\begin{aligned} d''(w) &= \tfrac{8K(\kappa)}{L}\left[\frac{-2cE(\kappa)K'(\kappa)+\kappa(2c-\eta_2^2)E(\kappa)K(\kappa)+2w^2\frac{d}{d\kappa}(K(\kappa)E(\kappa))}{2cK'(\kappa)-\kappa(2c-\eta_2^2)K(\kappa)}\right] \\ &= \tfrac{8K(\kappa)}{L}\left[\frac{G(\kappa)+w^2F(\kappa)}{c((2-\kappa^2)E(\kappa)-2(1-\kappa^2)K(\kappa))}\right], \end{aligned} \tag{34}$$

where

$$F(\kappa) = \left[2(2-\kappa^2)E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)(1-\kappa^2)K^2(\kappa)\right]$$
$$G(\kappa) = 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)E^2(\kappa).$$

Since $c((2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)) > 0$, then the sign of $d''(w)$ depends on the sign of quantity $G(\kappa) + w^2 F(\kappa)$. We have that $F(\kappa) > 0$.

Figure 1. The function $-G(\kappa)/F(\kappa)$, $0 \leq \kappa \leq 1$

Thus, we have stability for $\kappa :\in (0,1)$ so that

$$|w| > \sqrt{-\frac{G(\kappa)}{F(\kappa)}} =: w_0(\kappa), \ \ \kappa \in (0,1)$$

We now need to further clarify the relationship between $L, \kappa$ and $\omega$ in order to have a complete proof of Theorem 2. More precisely, we need to construct the function $\omega_0(L)$, which we reference in the statement of Theorem 2. To recapitulate, we have shown that $L, \omega, \kappa$ are related in (23) and we have orbital stability for $|\omega| > \omega_0(\kappa) = \sqrt{-\frac{G(\kappa)}{F(\kappa)}}$. Solving the relationship (23) for $\omega$ yields

$$\sqrt{1-\omega^2} = \frac{2\sqrt{2-\kappa^2}K(\kappa)}{L} \tag{35}$$

Note that here $L$ is fixed and the function $\kappa \to \sqrt{2-\kappa^2}K(\kappa)$ is increasing, whence there is at most one $\omega \in (0,1)$, for every $\kappa \in (0,1)$ satisfying the relationship (35). Now, the orbital stability condition $|\omega| > \omega_0(\kappa)$ is equivalent to

$$\frac{4(2-\kappa^2)K^2(\kappa)}{L^2} = 1 - \omega^2 \leq 1 - \omega_0^2(\kappa) = 1 + \frac{G(\kappa)}{F(\kappa)},$$

which is equivalent to the inequality

$$h[\kappa] := \frac{4(2-\kappa^2)K^2(\kappa)F(\kappa)}{F(\kappa) + G(\kappa)} \leq L^2 \tag{36}$$

As it can be seen from the graph of the function $h$, it is an increasing function. Moreover, using Mathematica, we have computed $\lim_{\kappa\to 0} h[\kappa] = 5\pi^2$. Therefore, the inequality (36) does not have solutions for $L < \sqrt{5}\pi$. For $L \geq \sqrt{5}\pi$, there is an increasing in $L$ function, $\kappa_0(L)$ (namely the inverse of $\kappa \to \sqrt{h[\kappa]}$), so that $0 \leq \kappa \leq \kappa_0(L)$ gives the solution to (36). Now, we may write the $\omega$ range of indices for which we have guaranteed orbital stability as follows

$$\frac{2\sqrt{2}K(0)}{L} \leq \sqrt{1-\omega^2} \leq \frac{2\sqrt{2-\kappa_0^2(L)}K(\kappa_0(L))}{L}$$

Taking into account $K(0) = \pi/2$ and $2\sqrt{2-\kappa_0^2(L)}K(\kappa_0(L)) = L\sqrt{1+\frac{G(\kappa_0(L))}{F(\kappa_0(L))}}$, yields

$$\sqrt{-\frac{G(\kappa_0(L))}{F(\kappa_0(L))}} \leq |\omega| \leq \sqrt{1-\frac{2\pi^2}{L^2}}.$$

Figure 2. The function $h[\kappa]$, $0 \leq \kappa \leq 0.7$

Several obvious corollaries from this analysis are in order. For $L < \sqrt{5}\pi$, our criteria does not provide orbital stability for *any* $\omega$ - basically the inequality above does not have solutions.

For $L > 2\pi$, we have that $\sqrt{1-\frac{2\pi^2}{L^2}} \geq \sqrt{\frac{1}{2}}$ and since $Ran(-G(\cdot)/F(\cdot)) \in (\frac{1}{2}, \frac{3}{5})$, we clearly have orbital stability at least for some interval $|\omega| \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}+\varepsilon)$. Finally, from (36), it is easy to see that

$$\lim_{L\to\infty} \kappa_0(L) = 1.$$

As a consequence

$$\lim_{L\to\infty}\sqrt{-\frac{G(\kappa_0(L))}{F(\kappa_0(L))}}=\frac{1}{\sqrt{2}},$$

whence at the limiting case of $L=\infty$, we have orbital stability for all $\omega:|\omega|\geq\frac{\sqrt{2}}{2}$, thus recovering Chen's result, [1].

## 4. Orbital stability for the periodic standing waves of the beam equation

We first address the existence statement in Proposition 2.

4.1. **Proof of Proposition 2.** We need to show the existence of solutions to (7). It is quite obvious, at least formally, to identify (7) as an Euler-Lagrange equation for certain minimization problem. Namely, consider the following minimization problem

$$\begin{aligned} I_\omega(u) &= \int_{[-L,L]^d}(|\Delta u(x)|^2+(1-\omega^2)|u(x)|^2)dx\to\min \\ \textit{subject to}\ \ K(u) &= \int_{[-L,L]^d}|u|^{p+1}dx=1 \end{aligned}$$

where $\omega\in(-1,1)$. The first thing to notice is that since $p<\frac{2d}{d-4}-1$, when $p\geq 5$, we have by Sobolev embedding

$$\|u\|_{L^{p+1}([-L,L]^d)}\leq C_{p,d}\|f\|_{H^2([-L,L]^d)}.$$

Therefore, $I_\omega(u)$ is bounded from below (by say $\frac{1-\omega^2}{C_{p,d}^2}$) for each admissible $u$. We conclude that the quantity

$$I_\omega^{\min}:=\inf_{\|u\|_{L^{p+1}}=1}I_\omega(u)>0,$$

is well-defined. Furthermore, we may take a smooth minimizing sequence $u_n$. That is, $\|u_n\|_{L^{p+1}}=1$, and

$$I_\omega(u_n)\to I_\omega^{\min}.$$

In particular, we have that $\sup_n\|u_n\|_{H^2}<\infty$. We first take an $H^2$ weakly convergent subsequence, denoted again by $u_n$, $u_n\to u$. By the compactness of the embedding $H^2([-L,L]^d)\hookrightarrow L^{p+1}([-L,L]^d)$, we can select a convergent (in the topology of $L^{p+1}([-L,L]^d)$) subsequence, let us denote it again by $u_n$, $u_n\to u$. Clearly $u:\|u\|_{L^{p+1}}=1$ and by the lower-semi continuity of the norms with respect to weak convergence, we have

$$I_\omega(u)\leq\liminf_n I_\omega(u_n)=I_\omega^{\min},$$

whence $u$ is an actual solution of the minimization problem.

We now apply the standard Euler-Lagrange scheme to derive that $u$ must solve (up to a coefficient) (7). More precisely, since $u$ is a minimizer, we have that for every test function $\chi\in H^\infty([-L,L]^d)$ and every sufficiently small $\varepsilon$,

$$I_\omega\left(\frac{u+\varepsilon\chi}{\|u+\varepsilon\chi\|_{L^{p+1}}}\right)\geq I_\omega(u)=I_\omega^{\min}.$$

Since $I_\omega\left(\frac{u+\varepsilon\chi}{\|u+\varepsilon\chi\|_{L^{p+1}}}\right)=\frac{I_\omega(u+\varepsilon\chi)}{\|u+\varepsilon\chi\|_{L^{p+1}}^2}$ and

$$\begin{aligned} &\|u+\varepsilon\chi\|_{L^{p+1}}=1+\varepsilon\langle|u|^{p-1}u,\chi\rangle+O(\varepsilon^2), \\ &I_\omega(u+\varepsilon\chi)=I_\omega^{\min}+2\varepsilon\langle\Delta u,\Delta\chi\rangle+2\varepsilon(1-\omega^2)\langle u,\chi\rangle+O(\varepsilon^2), \end{aligned}$$

we arrive at

$$I_\omega^{\min} \le I_\omega\left(\frac{u+\varepsilon\chi}{\|u+\varepsilon\chi\|_{L^{p+1}}}\right) = I_\omega^{\min} + 2\varepsilon(\langle \Delta u, \Delta\chi\rangle + (1-\omega^2)\langle u,\chi\rangle - I_\omega^{\min}\langle |u|^{p-1}u,\chi\rangle) + O(\varepsilon^2).$$

Clearly, since the last inequality has to be satisfied (for fixed $\chi$) for all sufficiently small $\varepsilon$, we get that

$$\langle \Delta u, \Delta\chi\rangle + (1-\omega^2)\langle u,\chi\rangle - I_\omega^{\min}\langle |u|^{p-1}u,\chi\rangle = 0,$$

for all test functions $\chi$. That is $u$ is a distributional solutions of the equation

$$\Delta^2 u + (1-\omega^2)u - I_\omega^{\min}|u|^{p-1}u = 0.$$

Setting $\varphi = (I_\omega^{\min})^{\frac{1}{p-1}}u$ produces a distributional solutions of (7). We have already shown that $\varphi \in H^2([-L,L]^d)$. Standard elliptic theory and bootstrapping arguments imply that such a $\varphi \in H^\infty([-L,L]^d)$.

4.2. **Proof of Theorem 3.** Define first the function

$$M(w) = I_\omega^{\min} = \inf\{I_\omega(u) : K(u) = 1\}$$

for every $\omega \in [0,1)$ (and one can then view it as an even function in $(-1,1)$). It is easy to see that the function $M(\omega)$ is decreasing. Indeed, let $0 \le \omega_1 < \omega_2 < 1$. For a fixed $u \in L^{p+1}, u \ne 0$, with $K(u) = 1$, we have

$$I_{\omega_1}(u) = \int (|\Delta u|^2 + (1-\omega_1^2)|u|^2)dx > \int (|\Delta u|^2 + (1-\omega_2^2)|u|^2)dx = I_{\omega_2}(u),$$

whence $M(\omega_2) < M(\omega_1)$. One can check that (7) implies the relation

$$E'(\varphi) - \omega Q'(\varphi) = 0. \tag{37}$$

Consider the set of functions

$$S_\omega = \{\psi \in H^2 \ , \ I_\omega(\psi) = K(\psi) = \frac{2(p+1)}{p-1}d(\omega)\}$$

The function $d(\omega)$ in (8) is well defined. It is also easy to see that

$$d(\omega) = \frac{p-1}{2(p+1)}I_w(\varphi_\omega) = \frac{p-1}{2(p+1)}K(\varphi_\omega). \tag{38}$$

We will now show that $\omega \to d(\omega)$ is decreasing in $(0,1)$. Indeed, by (38), we have

$$d(\omega) = \frac{p-1}{2(p+1)}K(\varphi_\omega) = \frac{p-1}{2(p+1)}K(u_\omega M(\omega)^{\frac{1}{p-1}}) = \frac{p-1}{2(p+1)}M(\omega)^{\frac{p+1}{p-1}},$$

where in the last step, we have used that $u_\omega$ solves the constrained minimization problem and hence $K(u_\omega) = 1$. Clearly, by this formula, it follows that $\omega \to d(\omega)$ is increasing, since $M(\omega)$ is decreasing.

**Lemma 5.** *Suppose that $d''(\omega) > 0$. Then there exists $\varepsilon > 0$ such that for all $\mathbf{u} \in U_{\omega,\varepsilon}$ and $\varphi \in S_\omega$*

$$E(\mathbf{u}) - E(\varphi) - \omega(\mathbf{u})(Q(\mathbf{u}) - Q(\varphi)) \ge \frac{1}{4}|\omega(\mathbf{u}) - \omega|^2,$$

*where $\omega(\mathbf{u})$ is defined by $\omega(\mathbf{u}) = d^{-1}(\frac{p-1}{2(p+1)}K(\mathbf{u}))$ and*

$$U_{\omega,\varepsilon} = \{\mathbf{u} \in X = H^2 \times L^2 \ : \ \inf\{||\mathbf{u} - \psi||_X \ : \ \psi \in S_\omega\} < \varepsilon\}$$

.

*Proof.* We have

$$\begin{aligned} E(\mathbf{u})-\omega(\mathbf{u})Q(\mathbf{u}) &= \tfrac{1}{2}I_\omega(\mathbf{u})-\tfrac{1}{p+1}K(\mathbf{u})+\tfrac{1}{2}\int|v-i\omega u|^2dx \\ &\geq \tfrac{1}{2}I_\omega(\mathbf{u})-\tfrac{1}{p+1}K(\mathbf{u}). \end{aligned}$$

Since

$$K(\mathbf{u})=\frac{2(p+1)}{p-1}d(\omega(\mathbf{u})),\quad K(\phi_{\omega(\mathbf{u})})=\frac{2(p+1)}{p-1}d(\omega(\mathbf{u}))$$

we have

$$K(\mathbf{u})=K(\varphi_{\omega(\mathbf{u})}),\quad I_{\omega(\mathbf{u})}(\mathbf{u})\geq I_{\omega(\mathbf{u})}(\varphi_{\omega(\mathbf{u})}).$$

From the above inequalities, we get

$$E(\mathbf{u})-\omega(\mathbf{u})Q(\mathbf{u})\geq d(\omega(\mathbf{u})).$$

From Taylor's expansion, we have (for $\omega$ sufficiently close to $\omega(\mathbf{u})$)

$$d(\omega(\mathbf{u}))\geq d(\omega)+d'(\omega)(\omega(\mathbf{u})-\omega)+\frac{1}{4}d''(\omega)|\omega(\mathbf{u})-\omega|^2.$$

Finally using that $d'(\omega)=Q(\mathbf{u})$, we have

$$E(\mathbf{u})-E(\varphi)-\omega(\mathbf{u})(Q(\mathbf{u})-Q(\varphi))\geq\frac{1}{4}d''(\omega)|\omega(\mathbf{u})-\omega|^2.$$

We will now show that if $d''(\omega)>0$, then $S_\omega$ is stable.

Assume the opposite for a contradiction, that is $S_w$ is unstable. Choose initial data $u_k(0)\in U_{w,\frac{1}{k}}$. Since $u_k(t)$ is continuous in $t$, we can find $t_k$ and $\psi_k\in S_w$, such that

$$\inf_{\psi\in S_w}||\mathbf{u}_k(t_k)-\psi_k||=\delta \tag{39}$$

and

$$|E(\mathbf{u}_k(t_k))-E(\psi_k)|<\frac{C}{k}$$
$$|Q(\mathbf{u}_k(t_k))-Q(\psi_k)|<\frac{C}{k}$$

From Lemma 5, we can choose $\delta$ so small such that

$$E(\mathbf{u}_k(t_k))-E(\psi_k)-\omega(\mathbf{u}_k(t_k))(Q(\mathbf{u}_k(t_k))-Q(\psi_k))\geq\frac{1}{4}|\omega(\mathbf{u}_k(t_k))-\omega|^2.$$

It follows that $w(u_k(t_k))\to w$ and

$$\lim_{k\to\infty}K(\mathbf{u}_k(t_k))=\frac{2(p+1)}{p-1}d(w)$$
$$\limsup_{k\to\infty}I_w(\mathbf{u}_k(t_k))\leq 2d(w)+\frac{4}{p-1}d(w)=\frac{2(p+1)}{p-1}d(w).$$

Hence

$$\lim_{k\to\infty}I_w(\mathbf{u}_k(t_k))=\frac{2(p+1)}{p-1}d(w)$$

and $M(w)^{\frac{1}{p-1}}u_k(t_k)$ is minimizing sequence and

$$\lim_{k\to\infty}||\mathbf{u}_k(t_k)-\phi_k||=0$$

which contradict (39).

□

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