Filomat 30:4 (2016), 1013–1020
DOI 10.2298/FIL1604013C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Weakly Second Spectrum of a Module

Seçil Çeken[a], **Mustafa Alkan**[b]

[a] *İstanbul Aydın University Department of Mathematics-Computer Programming İstanbul, Turkey*
[b] *Akdeniz University, Department of Mathematics, Antalya, Turkey*

Abstract. In this paper, we extend the definition of weakly second submodule of a module over a commutative ring to a module over an arbitrary ring. First, we investigate some properties of weakly second submodules. We define the notion of weakly second radical of a submodule and determine the weakly second radical of some modules. We also define the notion of weak m^*-system and characterize the weakly second radical of a submodule in terms of weak m^*-systems. Then we introduce and study a topology on the set of all weakly second submodules of a module. We give some results concerning irreducible subsets, irreducible components and compactness of this topological space. Finally, we investigate this topological space from the point of view of spectral spaces.

1. Introduction

Throughout this paper all rings will be associative rings with identity elements and all modules will be unital right modules. Unless otherwise stated R will denote a ring. By a proper submodule N of a non-zero right R-module M, we mean a submodule N with $N \neq M$. Given a right R-module M, we shall denote the annihilator of M (in R) by $ann_R(M)$.

A non-zero R-module M is called a *prime module* if $ann_R(M) = ann_R(K)$ for every non-zero submodule K of M. A proper submodule N of a module M is called a *prime submodule* of M if M/N is a prime module. The set of all prime submodules of a module M is called the *prime spectrum* of M and denoted by $Spec(M)$. Several authors investigated and topologized the prime spectrum of a given module (see [11], [12], [20]).

In [23], S. Yassemi introduced second submodules of modules over commutative rings as the dual notion of prime submodules. Second modules over arbitrary rings were defined in [2] and used as a tool for the study of attached primes over noncommutative rings. A right R-module M is called a *second module* provided $M \neq (0)$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M. By a *second submodule* of a module, we mean a submodule which is also a second module. If N is a second submodule of a module M, then $ann_R(N) = P$ is a prime ideal of R and in this case N is called a *P-second submodule* of M. Second submodules have been extensively studied in a number of papers (see for example [1], [5], [6], [14], [15], [16], [17]). The set of all second submodules of a module M is called the *second spectrum* of M and denoted by $Spec^s(M)$. Recently, some authors have investigated and topologized the second spectrum of a given module (see for example [1], [7], [8], [13], [18]).

2010 *Mathematics Subject Classification*. 16D10, 16D80
Keywords. Weakly second submodule, second submodule, weakly second radical, weakly second classical Zariski topology.
Received: 07 July 2015; Accepted: 28 September 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
The present investigation was supported by the Scientific Research Project Administration of Akdeniz University
Email addresses: cekenseci l@gmail.com (Seçil Çeken), alkan@akdeniz.edu.tr (Mustafa Alkan)

In [9], M. Behboodi and H. Koohy introduced the notion of weakly prime submodule as a generalization of prime submodule. A right R-module M is called *weakly prime module* if the annihilator of any non-zero submodule of M is a prime ideal and a proper submodule P of M is called *weakly prime submodule* if the quotient module M/P is a weakly prime module. In [4], H. Ansari-Toroghy and F. Farshadifar defined weakly second submodules of modules over commutative rings as the dual notion of weakly prime submodules. Let R be a commutative ring. A non-zero submodule N of an R-module M is called a weakly second submodule of M if $Nrs \subseteq K$, where $r, s \in R$ and K is a submodule of M, implies either $Nr \subseteq K$ or $Ns \subseteq K$. In this paper, we extend this definition to modules over arbitrary rings and then we introduce and study a topology on the set of weakly second submodules of a given module.

2. Weakly Second Submodules

Let R be a commutative ring. A non-zero submodule N of an R-module M is called a weakly second submodule of M if $Nrs \subseteq K$, where $r, s \in R$ and K is a submodule of M, implies either $Nr \subseteq K$ or $Ns \subseteq K$ (see [4, Definition 3.1]). We extend this definition to modules over arbitrary rings as follows.

Definition 2.1. *Let M be a non-zero right R-module. M is called a weakly second module if for all ideals A, B of R and for every submodule K of M, $MAB \subseteq K$ implies either $MA \subseteq K$ or $MB \subseteq K$. A submodule N of M is called a weakly second submodule of M if N is a weakly second module itself.*

Remark 2.2. (1) *It is clear that if R is a commutative ring, then Definition 2.1 is equivalent to [4, Definition 3.1]. Also if a submodule of a module over a noncommutative ring satisfies the commutative definition given in [4], then it satisfies Definition 2.1. But, if R is a noncommutative ring, these two definitions are not equivalent in general. For example, consider the matrix ring $R := M_n(k)$ of $n \times n$-matrices with entries in a field k and $n \geq 2$. It is well-known that R is a simple ring. Therefore, $M := R_R$ is a weakly second module in the sense of Definition 2.1, but it does not satisfy the commutative definition given in [4].*

(2) *It is easy to see that a non-zero right R-module M is weakly second if and only if $ann_R(M/N)$ is a prime ideal of R for every proper submoule N of M.*

(3) *It is easy to see that every non-zero factor module of a weakly second module is also weakly second.*

(4) *It is clear that every second module is weakly second.*

The following example shows that a weakly second module over a noncommutative ring need not to be a second module.

Recall that a fully prime ring is a ring with the property that every proper ideal is prime (see [22]). Clearly every non-zero module over a fully prime ring is weakly second.

Example 2.3. *Let F be a field and let R be the set of infinite matrices over F that have the form*

$$\begin{bmatrix} A & 0 & 0 & 0 & \dots \\ 0 & a & 0 & 0 & \dots \\ 0 & 0 & a & 0 & \dots \\ 0 & 0 & 0 & a & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where A is an arbitrary finite matrix and a is any element of F. In [22, Example 3.2], it was shown that R is a fully prime ring, R has only one non-zero proper ideal P and $P^2 = P$. Thus R_R is a weakly second module which is not second.

Proposition 2.4. *The following statements are equivalent for a submodule N of a right R-module M.*

(1) *N is a weakly second submodule of M.*

(2) *$\{ann_R(N/K) : K \leq N\}$ is a chain of prime ideals of R.*

(3) *For ideals A, B of R, either $NAB = NA$ or $NAB = NB$.*

(4) *$ann_R(N/K)$ is a prime ideal of R for every proper submoule K of N.*

Proof. (1) $\Longrightarrow$ (2) Let N be a weakly second submodule of M and K, L be proper submodules of N. Then, $ann_R(N/K)ann_R(N/L) \subseteq ann_R(N/K \cap L)$. Since $ann_R(N/K \cap L)$ is a prime ideal, we have either $ann_R(N/K) \subseteq ann_R(N/K \cap L) \subseteq ann_R(N/L)$ or $ann_R(N/L) \subseteq ann_R(N/K \cap L) \subseteq ann_R(N/K)$.

(2) $\Longrightarrow$ (1) and (1) $\Longleftrightarrow$ (4) Clear by Remark 2.2-(2).

(1) $\Longleftrightarrow$ (3) Clear from the definition. □

Theorem 2.5. *Let R be a commutative ring and M be a Noetherian right R-module. Then every weakly second submodule of M is second.*

Proof. Let N be a weakly second submodule of M. N is isomorphic to a subdirect product of its cocyclic factor modules $\{M_\lambda\}_{\lambda\in\Lambda}$. For each $\lambda \in \Lambda$, M_λ is a finitely generated weakly second module. By [4, Proposition 3.4-(a)], M_λ is a weakly prime module, i.e., (0) is a weakly prime submodule of M_λ for each $\lambda \in \Lambda$. [10, Proposition 2.4] implies that (0) is an intersection of prime submodules of M_λ. Since M_λ is a cocyclic module, any intersection of non-zero submodules of M_λ is again non-zero. So M_λ is a prime module for every $\lambda \in \Lambda$. [14, Lemma 1.3] implies that M_λ is homogeneous semisimple and so $ann_R(M_\lambda)$ is a maximal ideal of R for each $\lambda \in \Lambda$. By Proposition 2.4, the set $\{ann_R(M_\lambda) : \lambda \in \Lambda\}$ is a chain. Since $ann_R(M_\lambda)$ is a maximal ideal for each $\lambda \in \Lambda$, the set $\{ann_R(M_\lambda) : \lambda \in \Lambda\}$ is singleton. On the other hand N is isomorphic to a submodule of $\prod_{\lambda\in\Lambda} M_\lambda$. This implies that $ann_R(N) = \cap_{\lambda\in\Lambda} ann_R(M_\lambda) = ann_R(M_\lambda)$, a maximal ideal of R. It follows that N is a second submodule of M. □

Let M be an R-module and N be a submodule of M. The sum of all second submodules of N is called the *second radical* of N and denoted by $sec(N)$. If there is no second submodule of N, then we define $sec(N) = 0$. N is called a *second radical submodule* in case $N = sec(N)$ (see [5] and [15]).

Now we define the notion of weakly second radical of a submodule.

Definition 2.6. *Let M be a right R-module and N be a submodule of M. The sum of all weakly second submodules of N is called the weakly second radical of N and denoted by w-$sec(N)$. If there is no weakly second submodule of N, then we define w-$sec(N) = (0)$. N is called a weakly second radical submodule if $N = w$-$sec(N)$.*

Remark 2.7. (1) *It is clear that $Soc(N) \subseteq sec(N) \subseteq w$-$sec(N)$ for a submodule N of a right R-module M where $Soc(N)$ is the sum of all simple submodule of N.*

(2) *Let M be a right R-module. A maximal weakly second submodule of M is a weakly second submodule L of M such that L is not properly contained in another weakly second submodule of M. It can be easily shown that whenever $\{S_i\}_{i\in I}$ is a chain of weakly second submodules of M, then $\cup_{i\in I} S_i$ is a weakly second submodule. By using this fact and applying Zorn's Lemma, one can see that every weakly second submodule of M is contained in a maximal weakly second submodue. This implies that if M has a weakly second submodule, then the weakly second radical of M is the sum of all maximal weakly second submodules.*

Theorem 2.8. *Let M be a right R-module. If M satisfies descending chain condition on weakly second radical submodules, then every non-zero submodule of M has only a finite number of maximal weakly second submodules.*

Proof. The proof is similar to the proof of [4, Theorem 3.9], hence omitted. □

Recall that an R-module M is said to be a *comultiplication module* if for any submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ (see [3]).

Proposition 2.9. *Let M be a right R-module. Then the following are true.*

(1) *Let M be a comultiplication module. Then, a submodule N of M is second if and only if it is weakly second. Consequently, w-$sec(M) = sec(M)$.*

(2) *If R is a ring such that every prime ideal of R is maximal, then every weakly second submodule of M is second, consequently w-$sec(M) = sec(M)$. In particular, if R is a right perfect ring, then w-$sec(M) = sec(M) = Soc(M)$.*

(3) *If R is a ring such that the ring R/P is right or left Goldie for every prime ideal P of R and M is an injective right R-module, then w-$sec(M) = sec(M)$.*

(4) *If R is a commutative ring and M is a Noetherian R-module, then w-$sec(M) = sec(M) = Soc(M)$.*

Proof. (1) Let N be a weakly second submodule of the comultiplication module M. Then N is also a comultiplication module by [3, Theorem 3.17-(d)] and $ann_R(N)$ is a prime ideal of R by Remark 2.2-(2). [1, Proposition 3.17] implies that N is a second submodule. The converse and the consequence are clear.

(2) Let N be a weakly second submodule of M. Then, the set $\{ann_R(N/K) : K \lesssim N\}$ is a chain of prime ideals. Since every prime ideal of R is maximal, this set is singleton. This means that N is a second submodule of M. Thus $w\text{-}sec(M) = sec(M)$. It is well-known that every prime ideal is maximal in a right perfect ring. So, in particular, if R is a right perfect ring $w\text{-}sec(M) = sec(M) = Soc(M)$ by [14, Corollary 1.4].

(3) Let N be a weakly second submodule of M. Then $p := ann_R(N)$ is a prime ideal of R. By [21, Proposition 2.27], $(0 :_M p)$ is a non-zero injective right (R/p)-module. [14, Corollary 2.7 and Corollary 2.4] implies that $(0 :_M p)$ is a second R-submodule of M. Since $N \subseteq (0 :_M p)$, we have $w\text{-}sec(M) \subseteq sec(M)$. The converse inclusion is always true. Thus $w\text{-}sec(M) = sec(M)$.

(4) This follows from Theorem 2.5 and [15, Theorem 3.1]. □

In the following example we show that in general $w\text{-}sec(M) \neq sec(M)$ for a module M.

Example 2.10. *Consider the ring R and the ideal P in Example 2.3. Let $M := R_R$. In Example 2.3, we have shown that M is a weakly second R-module. Hence $w\text{-}sec(M) = M$. We claim that $sec(M) = P$. Since P is idempotent and it is the only non-zero proper ideal of R, P is a second submodule of M. Let I be a second submodule of M. Then either $IP = (0)$ or $IP = I$. If $IP = (0)$, then we get that $I = (0)$ as R is a prime ring. Thus we must have $I = IP = I \cap P$ and hence $I \subseteq P$. This implies that $sec(M) = sec(R_R) = P$. Thus $w\text{-}sec(M) \neq sec(M)$.*

Now we define the notion of weak m^*-system and characterize the weakly second radical of submodules in terms of weak m^*-systems.

Definition 2.11. *Let M be a right R-module. A subset $S \subsetneq M\backslash\{0\}$ is called a weak m^*-system, if $(K :_L A) \cup S \neq M$ and $(K :_L B) \cup S \neq M$, where A, B are ideals of R and K, L are submodules of M, then $(K :_L AB) \cup S \neq M$.*

Proposition 2.12. *Let M be a right R-module. A submodule Q of M is weakly second if and only if $M\backslash Q$ is a weak m^*-system.*

Proof. Suppose that Q is a weakly second submodule of M and $S := M\backslash Q$. Let A, B be ideals of R and K, L be submodules of M such that $(K :_L A) \cup S \neq M$ and $(K :_L B) \cup S \neq M$. Assume that $(K :_L AB) \cup S = M$. Then $QAB \subseteq K \cap L$. Since Q is weakly second, we have either $QA \subseteq K \cap L$ or $QB \subseteq K \cap L$. It follows that $(K :_L A) \cup S = M$ or $(K :_L B) \cup S = M$, a contradiction. Therefore S is a weak m^*-system.

Conversely, suppose that $S := M\backslash Q$ is a weak m^*-system. Let $QAB \subseteq K$, where A, B are ideals of R and K is a submodule of Q. Assume that $QA \nsubseteq K$ and $QB \nsubseteq K$. Then $(K :_M A) \cup S \neq M$ and $(K :_M B) \cup S \neq M$. Since S is a weak m^*-system, we have $(K :_M AB) \cup S \neq M$. This implies that $QAB \nsubseteq K$, a contradiction. Thus Q is a weakly second submodule of M. □

Proposition 2.13. *Let M be a right R-module, $S \subsetneq M\backslash\{0\}$ be a weak m^*-system and Q be a submodule of M minimal with the property that $Q \cup S = M$. Then Q is a weakly second submodule of M.*

Proof. Let $QAB \subseteq K$, where A, B are ideals of R and K is a submodule of Q. Assume that $QA \nsubseteq K$ and $QB \nsubseteq K$. By the minimality of Q, we have $(K :_Q A) \cup S \neq M$ and $(K :_Q B) \cup S \neq M$. This implies that $(K :_Q AB) \cup S \neq M$, and so $Q \cup S \neq M$, a contradiction. Therefore Q is a weakly second submodule of M. □

Definition 2.14. *Let M be a right R-module. For a submodule N of M, if there is a weakly second submodule of N, then we define*

$$\sqrt[ws]{N} := \{x \in N : \text{there is a weak } m^*\text{-system } S \text{ such that } x \notin S \text{ and } N \cup S = M\}$$

If there is no weakly second submodule of N, then we put $\sqrt[ws]{N} = (0)$.

Theorem 2.15. *Let M be a right R-module and $N \leq M$. Then $\sqrt[ws]{N} = w\text{-}sec(N)$.*

Proof. Suppose that $\sqrt[ws]{N} \neq (0)$. Let $x \in \sqrt[ws]{N}$. Then there is a weak m^*-system S such that $x \notin S$ and $N \cup S = M$. Let $\Psi = \{P \subseteq N : P \cup S = M\}$. Then $\Psi \neq \varnothing$, as $N \in \Psi$. Ψ is partially ordered with respect to reverse inclusion. Let $\{Q_i\}_{i\in\Lambda}$ be a chain in Ψ. It is clear that $\cap_{i\in\Lambda} Q_i \in \Psi$ is an upper bound for Ψ. By Zorn's Lemma, Ψ has a minimal element Q with respect to inclusion. By Proposition 2.13, Q is a second submodule of N and we have $x \in Q$. Thus $\sqrt[ws]{N} \subseteq sec(N)$.

Let Q be a second submodule of N. By Proposition 2.12, $S = M \backslash Q$ is a weak m^*-system. We also have $N \cup S = M$ and $x \notin S$ for every $x \in Q$. Therefore $Q \subseteq \sqrt[ws]{N}$ and so $w\text{-}sec(N) \subseteq \sqrt[ws]{N}$. □

3. The Weakly Second Classical Zariski Topology of a Module

Let M be a right R-module. The set of all weakly second submodules of M will be called the weakly second spectrum of M and denoted by $Spec^{ws}(M)$. In this section we introduce and study a topology on $Spec^{ws}(M)$.

Let M be a non-zero right R-module. For any submodule N of M we define the set $V^{ws}(N) := \{S \in Spec^{ws}(M) : S \subseteq N\}$. Then

(*i*) $V^{ws}(M) = Spec^{ws}(M)$ and $V^{ws}(0) = \emptyset$.

(*ii*) $\cap_{i\in I} V^{ws}(N_i) = V^{ws}(\cap_{i\in I} N_i)$.

(*iii*) $V^{ws}(N) \cup V^{ws}(L) \subseteq V^{ws}(N + L)$.

Let $WS(M) := \{V^{ws}(N) : N \leq M\}$. Then $WS(M)$ contains the empty set and $Spec^{ws}(M)$, and also $W^{ws}(M)$ is closed under arbitrary intersections. However, in general, $WS(M)$ is not closed under finite unions. For example, consider the $\mathbb{Z}$-module $M := \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Theroem 2.5, $Spec^{ws}(M) = Spec^{s}(M)$. [7, Remark 2.8] implies that $WS(M)$ is not closed under finite unions.

Definition 3.1. *Let M be a right R-module. For each submodule N of M, we put $W^{ws}(N) = Spec^{ws}(M) \backslash V^{ws}(N)$ and $\Omega^{ws}(M) = \{W^{ws}(N) : N \leq M\}$. Then we define $\eta^{ws}(M)$ to be the topology on $Spec^{ws}(M)$ by the sub-basis $\Omega^{ws}(M)$. In fact, $\eta^{ws}(M)$ is the collection of all unions of finite intersections of elements of $\Omega^{ws}(M)$. We call this topology the weakly second classical Zariski topology of M.*

Let M be a right R-module. Note that the set

$$\{W^{ws}(N_1) \cap ... \cap W^{ws}(N_k) : N_i \leq M,\ 1 \leq i \leq k,\ k \in \mathbb{N}\}$$

is a basis for the weakly second classical Zariski topology of M.

For each subset Y of $Spec^{ws}(M)$, we will denote the closure of Y in $Spec^{ws}(M)$ by $cl(Y)$.

Proposition 3.2. *Let M be a right R-module. Then the following are true.*

(1) *If Y is a finite subset of $Spec^{ws}(M)$, then $cl(Y) = \cup_{S\in Y} V^{ws}(S)$.*

(2) *If Y is a closed subset of $Spec^{ws}(M)$, then $Y = \cup_{S\in Y} V^{ws}(S)$.*

(3) *$Spec^{ws}(M)$ is a T_0-space.*

Proof. The proofs of (1) and (2) are similar to the proofs of [8, 3.1-(a), (b)], hence omitted.

(3) This follows from part (1) and the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct. □

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topological space X is called an *irreducible subset* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$. A maximal irreducible subspace of X is called an *irreducible component of* X. An irreducible component of a topological space is necessarily closed. Every irreducible subset of X is contained in an irreducible component of X, whence X is the union of its irreducible components.

Let M be a right R-module and $Y \subseteq Spec^{ws}(M)$. We will denote $\sum_{S\in Y} S$ by $T^{w}(Y)$.

Theorem 3.3. *Let M be a right R-module and $Y \subseteq Spec^{ws}(M)$. Then the following are true.*
(1) *For every $S \in Spec^{ws}(M)$, $V^{ws}(S)$ is irreducible.*
(2) *If Y is irreducible, then $T^w(Y)$ is a weakly second submodule of M.*
(3) *If $T^w(Y)$ is a weakly second submodule of M and $T^w(Y) \in cl(Y)$, then Y is irreducible.*

Proof. (1) The proof is similar to the proof of [18, Lemma 2.2], hence omitted.

(2) Let Y be an irreducible subset of $Spec^{ws}(M)$. Then, clearly, $T^w(Y) \neq (0)$ and $Y \subseteq V^{ws}(T^w(Y))$. Let A, B be ideals of R and K be a submodule of $T^w(Y)$ such that $T^w(Y)AB \subseteq K$. Then we see that $Y \subseteq V^{ws}((K :_M AB)) \subseteq V^{ws}((K :_M A)) \cup V^{ws}((K :_M B))$. Since Y is irreducible, we have either $Y \subseteq V^{ws}((K :_M A))$ or $Y \subseteq V^{ws}((K :_M B))$. This implies that either $T^w(Y)A \subseteq K$ or $T^w(Y)B \subseteq K$. Thus $T^w(Y)$ is a weakly second submodule of M.

(3) Let $S := T^w(Y)$ be a weakly second submodule of M and $S \in cl(Y)$. By Proposition 3.2-(1), we have $cl(\{S\}) = V^{ws}(S) \subseteq cl(Y)$ and clearly $Y \subseteq V^{ws}(S)$. Thus $cl(Y) = V^{ws}(S)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed subset of $Spec^{ws}(M)$. Then we have $cl(Y) = V^{ws}(S) \subseteq Y_1 \cup Y_2$. By part (1), $V^{ws}(S)$ is irreducible. This implies that either $Y \subseteq Y_1$ or $Y \subseteq Y_2$. So Y is irreducible. □

Corollary 3.4. *Let M be a right R-module and N be a submodule of M. Then $V^{ws}(N)$ is irreducible if and only if w-$sec(N)$ is a weakly second submodule of M. Consequently, $Spec^{ws}(M)$ is irreducible if and only if w-$sec(M)$ is a weakly second submodule of M.*

Proof. Suppose that $Y := V^{ws}(N)$ is irreducible. Then $T^w(Y) = w$-sec(N) is a weakly second submodule of M by Theorem 3.3-(2). Conversely, suppose that w-$sec(N)$ is a weakly second submodule of M. Then, by Proposition 3.2-(2), $Y := V^{ws}(N) = \cup_{S \in Y} V^{ws}(S)$. So w-$sec(N) \in cl(Y)$. Hence $V^{ws}(N)$ is irreducible by Theorem 3.3-(3). □

Theorem 3.5. *Let M be a right R-module such that $Spec^{ws}(M) \neq \emptyset$. Then, there is a bijective correspondence between the set of all maximal weakly second submodules of M and the set of irreducible components of $Spec^{ws}(M)$.*

Proof. Let $Max^{ws}(M)$ denote the set of all maximal weakly second submodules of M and let $IC(Spec^{ws}(M))$ denote the set of all irreducible components of $Spec^{ws}(M)$. Let $S \in Max^{ws}(M)$. Then $V^{ws}(S)$ is an irreducible closed subset of $Spec^{ws}(M)$ by Proposition Theorem 3.3-(1). Suppose that A is an irreducible subset of $Spec^{ws}(M)$ such that $V^{ws}(S) \subseteq A$. Then $S \subseteq T^w(A)$ and $T^w(A)$ is a weakly second submodule of M by Theorem 3.3-(2). By the maximality of S, we must have $S = T^w(A)$. This implies that $A = V^{ws}(S)$ and this shows that $V^{ws}(S)$ is an irreducible component of $Spec^{ws}(M)$. Thus we can define the map

$$\psi : Max^{ws}(M) \longrightarrow IC(Spec^{ws}(M)) \text{ by } \psi(S) = V^{ws}(S) \text{ for every } S \in Max^{ws}(M).$$

Clearly, ψ is well-defined and one to one. Now we show that ψ is surjective. Let $Y \in IC(Spec^{ws}(M))$. Then $T^w(Y)$ is a weakly second submodule of M by Theorem 3.3-(2). So there is a maximal weakly second submodule K of M such that $T^w(Y) \subseteq K$. It follows that $V^{ws}(T^w(Y)) \subseteq V^{ws}(K)$ and so $Y \subseteq V^{ws}(K)$. Since $V^{ws}(K)$ is irreducible by Theorem 3.3-(1), the maximality of Y implies that $Y = V^{ws}(K) = \psi(K)$. Thus ψ is a bijective map. □

Corollary 3.6. *Let M be an Artinian right R-module, then $Spec^{ws}(M)$ has only a finite number of irreducible components.*

Proof. The assertion follows from Theorem 3.5 and Theorem 2.8. □

A topological space X is said to be a *spectral space* if X is homeomorphic to $Spec(S)$, with the Zariski topology, for some commutative ring S. Spectral spaces were characterized by Hochster [19, p. 52, Proposition 4] as the topological spaces X which satisfy the following conditions:
(a) X is a T_0-space;
(b) X is compact and has a basis of compact open subsets;
(c) The family of compact open subsets of X is closed under finite intersections;
(d) Every irreducible closed subset of X has a generic point.

In the rest of this section we will prove that if M is a right R-module such that M satisfies descending chain condition on weakly second radical submodules, then $Spec^{ws}(M)$ is a spectral space.

Definition 3.7. *Let M be a right R-module, and let $P(M)$ be the family of all subsets of $Spec^{ws}(M)$ of the form $V^{ws}(N) \cap W^{ws}(K)$, where $N, K \leq M$. Clearly, $P(M)$ contains $Spec^{ws}(M)$ and $\emptyset$. Let $Z^{w}(M)$ be the collection of all unions of finite intersections of elements of $P(M)$. Then $Z^{w}(M)$ is a topology on $Spec^{ws}(M)$ and is called the finer patch-like topology of M. In fact, $P(M)$ is a sub-basis for the finer patch-like topology of M.*

Theorem 3.8. *Let M be a right R-module such that M satisfies descending chain condition on weakly second radical submodules. Then $Spec^{ws}(M)$ with the finer patch-like topology is a compact space.*

Proof. Let A be a family of finer patch-like open sets covering $Spec^{ws}(M)$, and suppose that no finite subfamily of A covers $Spec^{ws}(M)$. Consider the set $\Psi = \{K \leq M : K = w\text{-}sec(K)$ and no finite subfamily of A covers $V^{ws}(K)\}$. Since $V^{ws}(M) = V^{ws}(w\text{-}sec(M)) = Spec^{ws}(M)$, $w\text{-}sec(M) \in \Psi$ and hence $\Psi \neq \emptyset$. Since M satisfies descending chain condition on weakly second radical submodules, Ψ has a minimal element, say N. We claim that N is a weakly second submodule of M. Suppose on the contrary that N is not a weakly second submodule of M. Then there exist a submodule L of N and ideals I, J of R such that $NIJ \subseteq L$, $NI \nsubseteq L$ and $NJ \nsubseteq L$. Thus $(L :_N I) \subsetneq N$ and $(L :_N J) \subsetneq N$. By the minimality of N, there exists a finite subfamily A' of A that covers both $V^{ws}((L :_N I))$ and $V^{ws}((L :_N J))$. Let $S \in V^{ws}(N)$. Since $NIJ \subseteq L$, we have $SIJ \subseteq L$. Since S is weakly second, we have either $SI \subseteq L$ or $SJ \subseteq L$. Thus either $S \in V^{ws}((L :_N I))$ or $S \in V^{ws}((L :_N J))$. Therefore $V^{ws}(N) \subseteq V^{ws}((L :_N I)) \cup V^{ws}((L :_N J))$. This means that $V^{ws}(N)$ is covered with the finite subfamily A' of A, a contradiction. Hence N is a weakly second submodule of M.

Now choose $U \in A$ such that $N \in U$. This implies that N has a finer patch-like neighborhood $\cap_{i=1}^{n}[W^{ws}(K_i) \cap V^{ws}(N_i)]$ for some $K_i, N_i \leq M$, $n \in \mathbb{N}$ such that $\cap_{i=1}^{n}[W^{ws}(K_i) \cap V^{ws}(N_i)] \subseteq U$. We claim that for each i $(1 \leq i \leq n)$

$$N \in W^{ws}(K_i \cap N) \cap V^{ws}(N) \subseteq W^{ws}(K_i) \cap V^{ws}(N_i).$$

To see this assume that $S \in W^{ws}(K_i \cap N) \cap V^{ws}(N)$ so that $S \nsubseteq K_i \cap N$ and $S \subseteq N$. This implies that $S \in W^{ws}(K_i)$. Since $N \in V^{ws}(N_i)$ and $S \subseteq N$, we have $S \in V^{ws}(N_i)$. Hence we have:

$$N \in \cap_{i=1}^{n} [W^{ws}(K_i \cap N) \cap V^{ws}(N)] \subseteq \cap_{i=1}^{n} [W^{ws}(K_i) \cap V^{ws}(N_i)] \subseteq U.$$

Thus $[\cap_{i=1}^{n} W^{ws}(K_i')] \cap V^{ws}(N)$, where $K_i' := K_i \cap N \subsetneq N$, is a neighborhood of N with $[\cap_{i=1}^{n} W^{ws}(K_i')] \cap V^{ws}(N) \subseteq U$. Since $V^{ws}(K_i') = V^{ws}(w\text{-}sec(K_i'))$, we may assume that $K_i' = w\text{-}sec(K_i')$ for each i $(1 \leq i \leq n)$. Since $K_i' \subsetneq N$, for each i $(1 \leq i \leq n)$, $V^{ws}(K_i')$ can be covered by some finite subfamily A_i' of A. But

$$\begin{aligned} V^{ws}(N) &= V^{ws}(N) \cap [\left(\cap_{i=1}^{n} W^{ws}(K_i')\right) \cup \left(\cap_{i=1}^{n} W^{ws}(K_i')\right)^c] \\ &= \left(V^{ws}(N) \cap [\cap_{i=1}^{n} W^{ws}(K_i')]\right) \cup \left(V^{ws}(N) \cap [\cap_{i=1}^{n} W^{ws}(K_i')]^c\right) \\ &= \left(V^{ws}(N) \cap (\cap_{i=1}^{n} W^{ws}(K_i'))\right) \cup (V^{ws}(N) \cap (\cup_{i=1}^{n} V^{ws}(K_i'))) \end{aligned}$$

Hence $V^{ws}(N)$ can be covered by $A_1' \cup ... \cup A_n' \cup \{U\}$, contrary to our choice of N. Thus there exists a finite subfamily of A which covers $Spec^{ws}(M)$. Therefore, $Spec^{ws}(M)$ is compact with the finer patch-like topology of M. □

Proposition 3.9. *Let M be a right R-module such that M satisfies descending chain condition on weakly second radical submodules. Then*

(1) *Every irreducible closed subset of $Spec^{ws}(M)$ has a generic point.*

(2) *For each $n \in \mathbb{N}$, and submodules N_i $(1 \leq i \leq n)$ of M, $W^{ws}(N_1) \cap ... \cap W^{ws}(N_n)$ is a compact subset of $Spec^{ws}(M)$.*

(3) *Compact open subsets of $Spec^{ws}(M)$ are closed under finite intersections.*

Proof. (1) This assertion is proved by using Theorem 3.8 and Proposition 3.2-(2) similar to the proof of [8, Proposition 3.13].

(2) This assertion is proved by using Theorem 3.8 similar to the proof of [8, Theorem 3.15].

(3) This assertion is proved by using part (2) similar to the proof of [8, Corollary 3.16]. □

Theorem 3.10. *Let M be a right R-module such that M satisfies descending chain condition on weakly second radical submodules. Then $Spec^{ws}(M)$ is a spectral space with the weakly second classical Zariski topology.*

Proof. $Spec^{ws}(M)$ is a T_0-space by Proposition 3.2-(3). By Proposition 3.9-(2), $\beta = \{W^{ws}(N_1) \cap ... \cap W^{ws}(N_n) : n \in \mathbb{N}, N_i \leq M, 1 \leq i \leq n\}$ is a basis for $Spec^{ws}(M)$ with the property that each basis element is compact. Theorem 3.8 implies that $Spec^{ws}(M)$ is compact. The compact open subsets of $Spec^{ws}(M)$ is closed under finite intersections by Proposition 3.9-(3). Finally every irreducible closed subset of $Spec^{ws}(M)$ has a generic point by Proposition 3.9-(1). Thus $Spec^{ws}(M)$ is a spectral space by Hochster's characterization. □

Acknowledgement
The authors would like to thank the Scientific Technological Research Council of Turkey (TUBITAK) for funding this work through the project 114F381. The second author was supported by the Scientific Research Project Administration of Akdeniz University.The authors are deeply grateful to the referees for their useful pointed comments and suggestions.

References

[1] J. Y. Abuhlail, Zariski topologies for coprime and second submodules, Algebra Colloquium, (2014) in press.
[2] S. Annin, Attached primes over noncommutative rings, J. Pure Appl. Algebra, 212 (2008), no. 3, 510-521.
[3] H. Ansari-Toroghy and F. Farshadifar, The dual notion of multiplication modules, Taiwanese J. Math. 11 (2007), no. 4, 1189–1201.
[4] H. Ansari-Toroghy and F. Farshadifar, The dual notions of some generalizations of prime submodules, Comm. Algebra 39 (7), (2011), 2396-2416.
[5] H. Ansari-Toroghy, F. Farshadifar, On the dual notion of prime submodules, Algebra Colloq., 19 (1), (2012), 1109-1116.
[6] H. Ansari-Toroghy, F. Farshadifar, On the dual notion of prime submodules II, Mediterr. J. Math., 9 (2), (2012) 327-336.
[7] H. Ansari-Toroghy and F. Farshadifar, The Zariski topology on the second spectrum of a module, Algebra Colloquium, (2014) in press.
[8] H. Ansari-Toroghy, S. Keyvani and F. Farshadifar, On the second spectrum of a module (II), Bull. Malays. Math. Sci. Soc. (2014), in press.
[9] M. Behboodi, H. Koohy, Weakly prime modules, Vietnam J. Math. 32 (2) (2004) 185–195.
[10] M. Behboodi, On weakly prime radical of modules and semi-compatible modules, Acta Math. Hungar., 113 (2006), 239-250.
[11] M. Behboodi and M. R. Haddadi, Classical Zariski Topology of Modules and Spectral Spaces I, Inter. Electronic J. Alg. 4 (2008), 104-130.
[12] M. Behboodi and M. R. Haddadi, Classical Zariski Topology of Modules and Spectral Spaces II, Inter. Electronic J. Alg. 4 (2008), 131-148.
[13] S. Çeken and M. Alkan, Dual of Zariski topology for modules, Book Series: AIP Conference Proceedings, 1389 (1), (2011), 357-360.
[14] S. Çeken, M. Alkan and P. F. Smith, Second modules over noncommutative rings, Communications in Algebra, 41 (1), (2013), 83-98.
[15] S. Çeken, M. Alkan and P. F. Smith, The dual notion of the prime radical of a module, Journal of Algebra, 392 (2013), 265–275.
[16] S. Çeken, M. Alkan, On graded second and coprimary modules and graded secondary representations, Bull. Malays. Math. Sci. Soc. 38 (4) (2015) 1317–1330, DOI: 10.1007/s40840-014-0097-6.
[17] S. Çeken, M. Alkan, On second submodules, Contemporary Mathematics 634 (2015), 67–77.
[18] S. Çeken, M. Alkan, On the second spectrum and the second classical Zariski topology of a module, Journal of Algebra And Its Applications 14 (10) (2015), DOI: 10.1142/S0219498815501509.
[19] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43-60.
[20] C.P. Lu, The Zariski topology on the prime spectrum of a module, Houston J. Math. 25 (3) (1999) 417-432.
[21] D.W. Sharpe and P. Vamos, Injective modules, Cambridge University Press, London, 1972.
[22] H. Tsutsui, Fully prime rings, Comm. Algebra 24 (1996) 2981–2989.
[23] S. Yassemi, The dual notion of prime submodules, Arch. Math (Brno) 37 (2001), 273–278.