Almost Global Stability of Nonlinear Switched Systems With Time-Dependent Switching

Özkan Karabacak, Ayşegül Kıvılcım, and Rafael Wisniewski

***Abstract*—For a dynamical system, it is known that the existence of a Lyapunov density implies almost global stability of an equilibrium. It is then natural to ask whether the existence of multiple Lyapunov densities for a nonlinear switched system implies almost global stability, in the same way as the existence of multiple Lyapunov functions implies global stability for nonlinear switched systems. In this paper, the answer to this question is shown to be affirmative as long as switchings satisfy a dwell time constraint with an arbitrarily small dwell time. Specifically, as the main result, we show that a nonlinear switched system with a minimum dwell time is almost globally stable if there exist multiple Lyapunov densities that satisfy some compatibility conditions depending on the value of the minimum dwell time. This result can also be used to obtain a minimum dwell time estimate to ensure almost global stability of a nonlinear switched systems. In particular, the existence of a common Lyapunov density implies almost global stability for any arbitrary small minimum dwell time. The results obtained for continuous-time switched systems are based on some sufficient conditions for the almost global stability of discrete-time nonautonomous systems. These conditions are obtained using the duality between Frobenius–Perron operator and Koopman operator.**

***Index Terms*—Almost global stability, common Lyapunov density, minimum dwell time, multiple Lyapunov densities, nonlinear switched systems.**

I. Introduction

There exist many examples of dynamical systems (for example, see [1] and [2]) that are not globally stable but almost globally stable. For such systems, there is a nonempty set of initial states that do not converge to the origin, but this set is negligible, as it has zero Lebesgue measure. Almost global stability, namely, convergence of almost all solutions to an invariant set, has been first considered by Milnor [3] as a candidate of a useful notion of an attractor. Almost global stability has proved to be useful in the systems and control theory after the work of Rantzer [1]. Rantzer showed that almost global stability of an equilibrium can be verified by the existence of a density function, which is now called Lyapunov density by many researchers. Since then, Lyapunov densities have been used for the analysis of dynamical systems [4]–[12], for the design of control systems [13]–[16], [17]–[19] and for safety verification [20].

Two factors can be mentioned that motivates the research on almost global stability via Lyapunov densities: First, for nonlinear control systems of form $\dot{x} = F(x) + G(x)u$ with input u, almost global stability can be characterized as an optimization problem that is linear in the design variables, namely the Lyapunov density $\rho(x)$ and the product $\tilde{\rho}(x) := \rho(x)u(x)$ of the Lyapunov density and the input; whereas, classical state-feedback controller design via Lyapunov functions may lead to nonconvex optimization problems [1]. Second, for the cases where state space is not Euclidean, global stabilization of equilibria may not be possible and, therefore, almost global stabilization might be one of the best global properties to expect [21].

On the other hand, an important problem in this field that retards the use of Lyapunov densities in control applications should also be mentioned: As opposed to Lyapunov functions, Lyapunov densities do not directly provide forward-invariant sets; hence, they do not prevent undesired overshoots in solutions and, therefore, may need to be accompanied by other tools (such as barrier functions) that rule out such transients [20].

Various extensions of Rantzer's result on almost global stability via Lyapunov densities have appeared in the literature. To mention a few, Lyapunov densities have been considered for discontinuous vector fields (switched systems with state-dependent switching) [22], for smoothly time-varying systems [23], [24] and for discrete-time nonlinear stochastic systems [25]. However, to the best of our knowledge, almost global stability of nonlinear switched systems with time-dependent switching has not been considered in the literature yet, and is the subject of study of this paper.

Switched systems with time-dependent switching may arise as abstractions of switched systems with state-dependent switching [26, Subsection 1.1.2]. They also model switched control systems where switching is due to an external system [27]. Additionally, switched systems where switching rate is

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bounded, for example, due to the speed limit of communication in networked control systems, can be modeled as switched systems with time-dependent switching [26]. They can also be used to characterize the link between information rate and stability, to design a switching control [27], to control water storage in a drinking water supply network [28], and to guarantee the finite-time stability of a position servomechanism system [29] (for more applications, see the references in [26], [27], and [30]).

As the main results on the global stability of nonlinear switched systems are formulated in terms of multiple Lyapunov functions [30], it is natural to investigate the consequences of the existence of multiple Lyapunov densities for a nonlinear switched system. Consequently, we pose the following question: Does the existence of multiple Lyapunov densities implies almost global stability of a switched system?

In the sequel, we provide an affirmative answer to this question for systems with (arbitrary small) dwell time switching. Specifically, we provide a sufficient condition for almost global stability based on multiple Lyapunov densities and a minimum dwell time (see Theorem 1). This result provides an estimation of the minimum dwell time that guarantees almost global stability. In particular, if a common Lyapunov density exists, the switched system is almost globally stable for all values of $\tau_{\min} > 0$ (see Corollary 1).

To prove the above-mentioned results for a continuous-time switched system, we fix a switching signal leading to a time-varying system, discretize the time-varying system (with a fixed but arbitrarily small sampling time) giving rise to a discrete-time nonautonomous system, and finally lean upon the almost global stability of the latter. To this end, we obtain sufficient conditions for almost global stability of discrete-time nonautonomous systems (see Lemma 2), which may be of interest in its own right. For simplicity, we consider almost global stability of a common equilibrium, however all results in this paper also hold when the common equilibrium is replaced by a common compact invariant set.

The proof of our main result is based on linear transfer operators, Frobenius–Perron operator, and Koopman operator, which are used to capture the global dynamics of a system (see [31]–[33]). This approach was first used for almost global stability by Vaidya and Mehta in [6], where they give a sufficient condition for the almost global stability of an invariant set for discrete-time, autonomous systems with compact state space using a local attraction assumption. This result is extended in [11] to systems with noncompact state space without using any local stability assumption and in [12] to the problem of finite-time stability. Our results on almost global stability of discrete-time nonautonomous systems are in the spirit of [6], [11], and [12]. Finally, we point out that similar techniques that relate properties of discrete-time systems to continuous-time systems have appeared in the literature, for instance, in [34], for the stability of sampled-data nonlinear systems.

The outline of this paper is as follows. Almost global stability of continuous-time nonlinear switched systems is discussed in Section II, which also contains the main result of this paper. The proof of the main result is given in Section III. Section IV contains some remarks on the monotonicity of multiple Lyapunov densities and on how the presented theory generalizes some already known linear techniques [35].

Notation: $\mathbb{R}(\mathbb{Z}), \mathbb{R}_{>0}(\mathbb{Z}_{>0})$ and $\mathbb{R}_{\geq 0}(\mathbb{Z}_{\geq 0})$ denote the set of all, positive and nonnegative real numbers (integers), respectively. For $\mathbb{R}^n$, the vector space of real n-tuples, $\|\cdot\|$ denotes the Euclidean norm and m denotes the Lebesgue measure on $\mathbb{R}^n$. $\int \cdot \, d\mu(x)$ indicates Lebesgue integral with respect to measure μ, whereas for simplicity Lebesgue integral with respect to Lebesgue measure m is denoted as $\int \cdot \, dx$. $\mathbf{0} \in \mathbb{R}^n$ denotes the zero vector. $B_\varepsilon = \{x \in \mathbb{R} \mid \|x\| < \varepsilon\}$ is the open $\varepsilon-$ball around $\mathbf{0}$ and B_ε^c is the complement of B_ε. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable away from $\mathbf{0}$ meaning that it is Lebesgue integrable on B_ε^c for all $\varepsilon > 0$. For functions $f, g : \mathbb{R}^n \to \mathbb{R}$, f is said to be of the same order as g meaning that $f(x) = O(g(x))$, i.e., $\left|\frac{f(x)}{g(x)}\right|$ is bounded as $\|x\| \to \infty$. $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be nonsingular, if $m(f^{-1}(A)) = 0$ for every measurable set A with $m(A) = 0$. For a set $V \subset \mathbb{R}^n$, 1_V denotes the characteristic function of V. For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{D}f$ denotes the Jacobian of f and $\nabla \cdot f$ denotes the divergence of f. For $f : \mathbb{R}^n \to \mathbb{R}$, ∇f denotes the gradient of f. For a matrix A, A^{T} denotes the transpose of A. For symmetric matrices A and B, we use the notation $A < B$ ($A \leq B$) to mean that $B - A$ is positive (semi)definite. Finally, we use the phrases "almost all," "almost every," and "almost everywhere" in the sense of Lebesgue measure, namely, the set of points for which the argument fails is contained in a set of zero Lebesgue measure.

II. Almost Global Stability of Switched Systems

In this section, we present sufficient conditions for almost global stability of nonlinear switched systems.

Initially, we state some results on the almost global stability of autonomous systems using Lyapunov densities, not only for the sake of completeness but also for their use in showing the global existence of almost all solutions of switched systems. Consider the following ordinary differential equation:

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(\mathbf{0}) = \mathbf{0}$. The following proposition can be seen as a modified version of Rantzer's theorem for autonomous systems for which almost all solutions are known to exist for all positive times.

Proposition 1: (Adapted from [11, Th. 4.2]) Suppose that for almost every $x_0 \in \mathbb{R}^n$, a forward-complete solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (1) with $x(0) = x_0$ exists. Assume that there exists a nonnegative, continuously differentiable function $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ satisfying the following condition

$$\rho(x) \text{ is integrable away from } \mathbf{0}$$

$$\nabla \cdot (\rho f)(x) > 0 \text{ for almost all } x \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Then, almost all solutions of (1) converge to $\mathbf{0}$ as $t \to \infty$.

The following proposition is used to ensure the global existence of almost all solutions.

Proposition 2: Assume that there exists a nonnegative, continuously differentiable function $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ satisfying

$$(1 + \|f(x)\|)\rho(x) \text{ is integrable away from } \mathbf{0}$$

$$\nabla \cdot (\rho f)(x) > 0 \text{ for almost all } x \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Then, for almost every initial state $x_0 \in \mathbb{R}^n$, a forward-complete solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (1) with $x(0) = x_0$ exists and converges to $\mathbf{0}$ as $t \to \infty$.

Proof: Consider the time scaling $t_{\text{new}} = \int_0^t [1 + \|f(x(s))\|]\, ds$, under which the scaled solutions $x(t_{\text{new}})$ satisfy the scaled system $dx/dt_{\text{new}} = f_{\text{new}}(x)$, where $f_{\text{new}}(x) := f(x)/(1 + \|f(x)\|)$. Solutions of the scaled vector field f_{new} exist globally and they produce the same trajectories as $x(t)$ with the direction of time preserved (see [36, page 184]). Therefore, it is enough to show the convergence of almost all solutions of f_{new} to $\mathbf{0}$, since the convergence of trajectories $x(t)$ to a bounded set implies their existence for all $t \in \mathbb{R}_{\geq 0}$. This can be done by applying Proposition 1 with $\rho_{\text{new}}(x) := (1 + \|f(x)\|)\rho(x)$, noting that $\nabla \cdot (\rho_{\text{new}} f_{\text{new}})(x) = \nabla \cdot (\rho f)(x) > 0$. ■

Remark 1: Proposition 2 differs from Rantzer's original theorem in that it assumes the integrability of $(1 + \|f(x)\|)\rho(x)$ away from $\mathbf{0}$ instead of the integrability of $\|f(x)\|\rho(x)/\|x\|$ away from $\mathbf{0}$. We prefer the former condition as it implies the integrability of $\rho(x)$ away from $\mathbf{0}$, which is used in the proof of the main theorem for switched systems ahead.

Let us consider a nonlinear switched system given by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \sigma \in \mathcal{S}_\tau, \quad t \in [0, \infty). \tag{2}$$

Here, $\sigma : [0, \infty) \to \{1, \ldots, N\}$ is called a switching signal that is a right-continuous, piecewise constant function with finitely many discontinuities on any finite interval. $\mathcal{S}_\tau$ denotes the set of all switching signals satisfying $t_k - t_{k-1} \geq \tau$, $k \in \mathbb{Z}_{>0}$, where t_k denote the kth discontinuity point of σ ($t_0 = 0$ is assumed) and τ is called a minimum dwell time. We call each system given by $\dot{x} = f_p(x)$, for $p \in \{1, 2, \ldots, N\}$ a subsystem of (2). We assume that each subsystem $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $p \in \{1, 2, \ldots, N\}$ is continuously differentiable and share a common equilibrium at $\mathbf{0}$, namely $f_p(\mathbf{0}) = \mathbf{0}$.

Let us denote the value of $\sigma(t)$ for $t \in [t_{k-1}, t_k)$ by p_k. A switching signal can then be identified using these values as

$$\sigma(t) = ((p_1, \Delta t_1), (p_2, \Delta t_2), \ldots) \tag{3}$$

where $\Delta t_k = t_k - t_{k-1} \geq \tau$ is the operation time for the subsystem f_{p_k} on the kth constant operation of the switched system. In examples, we will mostly use periodic switching signals, which we identify by a finite sequence (showing the shortest repeating pattern) as

$$\begin{aligned} \sigma(t) &= ((p_1, \Delta t_1), \ldots, (p_n, \Delta t_n)) \\ &= ((p_1, \Delta t_1), \ldots, (p_n, \Delta t_n), (p_1, \Delta t_1), \ldots) \end{aligned} \tag{4}$$

which has a minimum period of $\Delta t_1 + \cdots + \Delta t_n$.

Definition 1: The nonlinear switched system (2) is said to be *almost globally stable for a* $\sigma \in \mathcal{S}_\tau$ if the following condition holds.

For almost every $x_0 \in \mathbb{R}^n$, a forward-complete solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (2) for the switching signal σ and the initial state $x(0) = x_0$ exists and converges to $\mathbf{0}$ as $t \to \infty$.

The system (2) is said to be *almost globally stable* if it is almost globally stable for every $\sigma \in \mathcal{S}_\tau$.

Note that if forward-complete solutions exist for almost all initial states for each subsystem, then forward-complete solutions of the switched system (2) exist for almost all initial states when $\sigma \in \mathcal{S}_\tau$ for some $\tau > 0$.

We are now ready to state our main result, which employs multiple Lyapunov densities and a dwell time condition to ensure almost global stability.

Theorem 1 (Main Result): Consider the switched system (2). Suppose that for each $p \in \{1, 2, \ldots, N\}$, there exist a constant $\kappa_p > 0$ and a nonnegative, continuously differentiable function $\rho_p : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, such that

$$(1 + \|f_p(x)\|)\rho_p(x) \text{ is integrable away from } \mathbf{0} \tag{5}$$

$$\nabla \cdot (\rho_p f_p)(x) \geq \kappa_p \rho_p(x) \quad \forall\, x \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{6}$$

Suppose also that the functions $\rho_p,\ p \in \{1, \ldots, N\}$ satisfy the following compatibility condition:

$$\begin{aligned} &\forall\, p, m \in \{1, \ldots, N\},\ \exists\, c_{pm} \in \mathbb{R}_{>0} : \\ &\rho_p(x) \leq c_{pm}\, \rho_m(x) \ \ \forall\, x \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \end{aligned} \tag{7}$$

Then, the system (2) is almost globally stable for any

$$\tau > \tau_{\min} := \min_{\beta_1, \ldots, \beta_N \in \mathbb{R}_{>0}} \ \max_{p, m \in \{1, 2, \ldots, N\}} \frac{\ln\left(\frac{\beta_p}{\beta_m} c_{pm}\right)}{\kappa_p}. \tag{8}$$

Proof: See Section III.

Remark 2: Once multiple Lyapunov densities satisfying the conditions (5)–(7) are found, computing $\tau_{\min}$ is not a difficult task. This is because the expression of $\tau_{\min}$ in (8) is equivalent to the so-called maximum cycle ratio of doubly weighted directed graphs [37] for which fast algorithms exist [38]. [1] In details, consider the globally coupled directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, w_1, w_2\}$, where $\mathcal{N} := \{1, \ldots, N\}$ is the set of vertices, $\mathcal{E} := \mathcal{N} \times \mathcal{N}$ is the set of directed edges, and the weights $w_1, w_2 : \mathcal{E} \to \mathbb{R}$ are defined as $w_1((p, m)) := \ln c_{pm}$ and $w_2((p, m)) = \kappa_p$. Let $\mathcal{C}$ be the set of all cycles in $\mathcal{G}$ and define the cycle ratio of a cycle $C = \{(p_0, p_1), \ldots, (p_{l-1}, p_l = p_0)\} \in \mathcal{C}$ of length l as

$$w(C) = \frac{\sum_{n=1}^{l} w_1((p_{n-1}, p_n))}{\sum_{n=1}^{l} w_2((p_{n-1}, p_n))}.$$

The *maximum cycle ratio* of $\mathcal{G}$ is then defined as $w_{\min} := \max_{C \in \mathcal{C}} w(C)$, which is equal to $\tau_{\min}$ by [37, Th. 1.1]. In particular, for bimodal systems, the dwell time condition (8) can be written as

$$\tau > \tau_{\min} := \frac{\ln c_{12} + \ln c_{21}}{\kappa_1 + \kappa_2}. \tag{9}$$

Let us consider the case where all Lyapunov densities considered in Theorem 1 are identical, namely there exists a common

[1] This also proves that the maximum in (8) is attained.

Lyapunov density satisfying conditions (5) and (6) and the condition (7) is satisfied for $c_{pm} = 1 \ \forall\, p, m$. In this case, $\tau_{\min}$ in (8) is obtained as zero by choosing $\beta_1 = \cdots = \beta_N$. Hence, we have the following corollary of Theorem 1 that shows that the existence of a common Lyapunov density implies almost global stability of a nonlinear switched system with an arbitrary small dwell time.

Corollary 1: Consider the switched system (2). Assume that there exist a constant $\kappa > 0$ and a nonnegative, continuously differentiable function $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ such that the following conditions are satisfied for all $p \in \{1, 2, \ldots, N\}$:

$$(1 + \|f_p(x)\|)\rho(x) \text{ is integrable away from } \mathbf{0} \tag{10}$$

$$\nabla \cdot (\rho f_p)(x) \geq \kappa \rho(x) \quad \forall\, x \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{11}$$

Then, the system (2) is almost globally stable for any $\tau > 0$.

Remark 3: It may be possible to generalize Corollary 1 to the set of arbitrary (nonchattering) switching signals $\mathcal{S}_{\text{nonchatt}}$, containing all signals that have finitely many discontinuities on finite intervals. Obviously, Corollary 1 applies only to $\cup_{\tau>0} S_\tau \subsetneq \mathcal{S}_{\text{nonchatt}}$. This is because our proof technique (see Section III) requires a minimum dwell time for each switching signal considered. Whether or not, Corollary 1 can be extended to $\mathcal{S}_{\text{nonchatt}}$ remains to be an open problem.

We now illustrate some applications of the main result. The following example shows that a nonlinear switched system with a common Lyapunov density is almost global stable but may not exhibits global stability.

Example 1: Consider the switched system (2) with $N = 3$ and the subsystems given as

$$f_1(x_1, x_2) = \begin{pmatrix} x_2 - x_1 + 3x_1x_2 \\ -x_2 - x_1 + x_2^2 - 2x_1^2 \end{pmatrix}$$

$$f_2(x_1, x_2) = \begin{pmatrix} -x_2 - x_1 + x_1^2 - 2x_2^2 \\ -x_2 + x_1 + 3x_1x_2 \end{pmatrix}$$

$$f_3(x_1, x_2) = \begin{pmatrix} x_2 - x_1 - x_1^2 + 2x_2^2 \\ -x_2 - x_1 - 3x_1x_2 \end{pmatrix}.$$

Let us consider $\rho(x) = \left(x_1^2 + x_2^2\right)^{-5/2}$. Equation (10) is satisfied because $(1 + \|f_p\|)\,\rho$ is of the same order as $\|x\|^{-3}$ for $p = 1, 2, 3$. Moreover, it can be shown that $\nabla \cdot (\rho f_p) = 3\rho$ for $p = 1, 2, 3$. Therefore, (11) is satisfied for $\kappa = 3$. As a result, by Corollary 1, $\mathbf{0}$ is almost globally stable.

Fig. 1 exhibits a solution of the system for the following periodic switching signal:

$$\sigma(t) = \big((1, 0.5), (2, 0.3), (3, 0.2)\big) \tag{12}$$

with period 1. It is seen in Fig. 1 that the backward solution (obtained by extending the switched system and the switching signal backward in time in a trivial way) approaches to a limit cycle as $t \to -\infty$, whereas the forward solution approaches to $\mathbf{0}$ as $t \to \infty$. The existence of the unstable limit cycle implies the lack of global stability for the switched system, i.e., not all initial states lead to convergence of solutions to the origin.

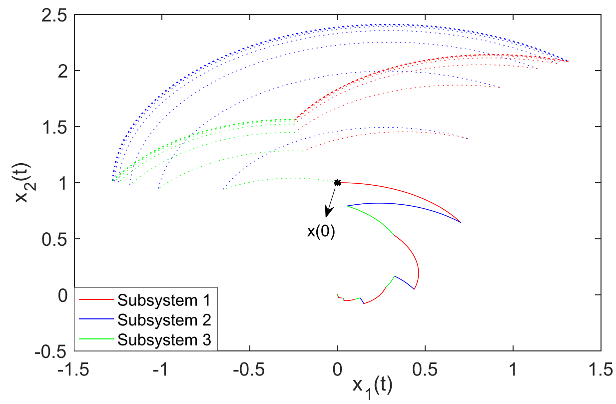

Fig. 1. Solution of Example 1. The dotted line is for the backward solution, which approaches to a limit cycle as $t \to -\infty$. The solid line is for the forward solution, which approaches to $\mathbf{0}$ as $t \to \infty$.

The following example illustrates an application of Theorem 1.

Example 2: Consider the switched system (2) with $N = 2$ and with subsystems

$$f_1(x_1, x_2) = \begin{pmatrix} -0.1x_1 + x_2 + 3x_1x_2 \\ -x_1 - 0.1x_2 - 2x_1^2 + x_2^2 \end{pmatrix}$$

$$f_2(x_1, x_2) = \begin{pmatrix} -0.1x_1 - 2x_2 + 0.5x_1^2 - 4x_2^2 \\ 0.5x_1 - 0.1x_2 + 1.5x_1x_2 \end{pmatrix}.$$

Let us consider $\rho_1(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{-5/2}$ and $\rho_2(x_1, x_2) = \left((0.5x_1)^2 + x_2^2\right)^{-5/2}$. Equation (5) is satisfied because $(1 + \|f_p\|)\,\rho_p$ is of the same order as $\|x\|^{-3}$ for $p = 1, 2$. Moreover, it can be shown that $\nabla \cdot (\rho_p f_p) = 0.3\rho_p$ for $p = 1, 2$. Therefore, (6) is satisfied for $\kappa_1 = \kappa_2 = 0.3$. Equation (7) is also satisfied since $\rho_1(x) \leq c_{12}\rho_2(x)$, for $c_{12} = 1$ and $\rho_2(x) \leq c_{21}\rho_1(x)$, for $c_{21} = 2^5$. Applying Theorem 1 with formula (9), we obtain that the origin is almost globally stable for all switching signals with dwell time $\tau > \tau_{\min} = \frac{\ln(2^5)}{0.6} = 5.7762$. Fig. 2 depicts a solution of the switched system for a periodic switching signal that does not satisfies the dwell time condition $\tau > 5.7762$.

III. Proof of the Main Result

The proof of the main result (see Theorem 1) is organized as follows: We first state a sufficient condition for the almost global stability of a discrete-time nonautonomous system (see Lemma 2). Then, the almost global stability of the continuous-time switched system (2) is characterized by the almost global stability of the discretizations of (2) for fixed switching signals (see Lemma 3). These lemmas, with the help of a monotonicity property of Lyapunov densities (see Lemma 5) result in a less conservative sufficient condition for the almost global stability of (2) (see Lemma 6), on which the proof of Theorem 1 is based.

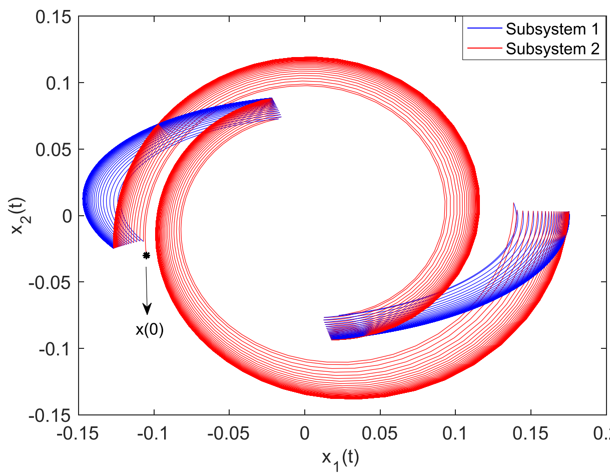

Fig. 2. Solution of Example 2 for the switching signal $\sigma(t) = ((1, 4.7), (2, 1.7))$ that does not satisfy the dwell time condition $\tau > 5.7762$. The solution approaches to a limit cycle.

A. Preliminaries for Transfer Operators

Let $\mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ denote the linear vector space of equivalence classes of measurable functions from $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to $\mathbb{R}$, where two functions are assumed to be equal if they agree on a set of full Lebesgue measure. Therefore, all statements for the functions in $\mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ should be understood to hold for almost all points in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For a nonsingular map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(\mathbf{0}) = \mathbf{0}$, let us denote the Frobenius–Perron operator and the Koopman operator for f restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as $\mathbb{P}$ and $\mathbb{U}$, respectively. Recall that $\mathbb{P} : \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \to \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ provides information about the evolution of densities (possibly with infinite mass) and is uniquely defined via

$$\int_V \mathbb{P}\rho dx = \int_{f^{-1}(V)} \rho dx \tag{13}$$

(see [39, p.238]). $\mathbb{P}$ is a positive operator, i.e., $\rho > 0 \Rightarrow \mathbb{P}\rho > 0$, and it can be written explicitly as

$$\mathbb{P}\rho(x) = \rho(f^{-1}(x)) \det \left(Df^{-1}\right) \tag{14}$$

whenever f is differentiable and invertible (see [31, Remark 3.2.4]). The Koopman operator $\mathbb{U} : \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \to \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ provides the evolution of observables (possibly essentially unbounded) and is defined as $\mathbb{U}g(x) = g(f(x))$. We define $\langle g, \rho\rangle := \int_{\mathbb{R}^n} g d\mu_\rho(x)$, where $\mu_\rho(V) := \int_V \rho dx$. Duality between $\mathbb{P}$ and $\mathbb{U}$ is expressed by $\langle \mathbb{U}g, \rho\rangle = \langle g, \mathbb{P}\rho\rangle$. Note that we allow both sides of the duality equation to be infinite. For more details on transfer operators, see [31] and [40].

B. Almost Global Stability of Discrete-Time Nonautonomous Systems

Consider the discrete-time nonautonomous system

$$x(k+1) = f_k\left(x(k)\right), \quad k \in \mathbb{Z}_{\geq 0} \tag{15}$$

where $f_k : \mathbb{R}^n \to \mathbb{R}^n, k \in \mathbb{Z}_{\geq 0}$, where f_ks are nonsingular maps. We assume that $\mathbf{0}$ is a common fixed point for all maps; namely, $f_k(\mathbf{0}) = \mathbf{0}, k \in \mathbb{Z}_{\geq 0}$. Denote the solution of (15) for an initial state $x(0) = x_0 \in \mathbb{R}^n$ by $\phi_k(x_0) = f_{k-1} \circ \cdots \circ f_0(x_0)$. We say that the system (15) is *almost globally stable* if $\lim_{k\to\infty} \phi_k(x) = \mathbf{0}$ for almost every $x \in \mathbb{R}^n$.

For maps f_k, $k \in \mathbb{Z}_{\geq 0}$, let us denote the Frobenius–Perron operator and the Koopman operator for f_k restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as $\mathbb{P}_k$ and $\mathbb{U}_k$, respectively. Similarly, Frobenius–Perron operators and Koopman operators for the solution maps ϕ_k, $k \in \mathbb{Z}_{>0}$ restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ can be written as

$$\mathbb{P}_{\to k} := \mathbb{P}_{k-1} \circ \cdots \circ \mathbb{P}_0$$
$$\mathbb{U}_{\to k} := \mathbb{U}_0 \circ \cdots \circ \mathbb{U}_{k-1}$$

which are dual to each other, namely $\langle g, \mathbb{P}_{\to k}\rho\rangle = \langle \mathbb{U}_{\to k} g, \rho\rangle$. We set $\mathbb{P}_{\to 0}$ and $\mathbb{U}_{\to 0}$ to be the identity operators.

The following result is a direct consequence of the Borel–Cantelli lemma.

Lemma 1: Equation (15) is almost globally stable if there exists a $\rho \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ such that $\rho(x) > 0$ and $\bar{\rho} := \sum_{k=0}^{\infty} \mathbb{P}_{\to k}\rho$ is integrable away from $\mathbf{0}$.[2]

Proof: For an $\varepsilon > 0$, consider the events $E_k^{(\varepsilon)} = \{x \in \mathbb{R}^n \mid \mathbb{U}_{\to k} 1_{B_\varepsilon^c}(x) = 1\} = f_0^{-1} \cdots f_{k-1}^{-1}(B_\varepsilon^c)$, where B_ε^c is the complement of the $\varepsilon-$ball of $\mathbf{0}$. Define $E^{(\varepsilon)} := \limsup_{k\to\infty} E_k^{(\varepsilon)} = \cap_{k=0}^{\infty} \cup_{k'=k}^{\infty} E_{k'}^{(\varepsilon)}$, which is the set of all initial states for which the solution of (15) visits B_ε^c infinitely often. It suffices to show that $m(E^{(\varepsilon)}) = 0$ for any $\varepsilon > 0$. This is because the set $\cup_k E^{(\varepsilon_k)}$ for a sequence $\varepsilon_k \to 0$ is a set of Lebesque measure zero and its complement contains the set of all initial state that converge to $\mathbf{0}$. Consider the measure $\mu_\rho(V) := \int_V \rho dx$, with respect to which m is absolutely continuous[3] (since $\rho(x) > 0$), i.e., $m(W) = 0$ whenever $\mu_\rho(W) = 0$. Therefore, we only need to show that $\mu_\rho(E^{(\varepsilon)}) = 0$. Note that $\mu_\rho(E_k^{(\varepsilon)}) = \langle \mathbb{U}_{\to k} 1_{B_\varepsilon^c}, \rho\rangle = \langle 1_{B_\varepsilon^c}, \mathbb{P}_{\to k}\rho\rangle$. Since $\bar{\rho}$ is integrable away from $\mathbf{0}$, we have $\langle 1_{B_\varepsilon^c}, \bar{\rho}\rangle < \infty$, and

$$\sum_{k=0}^{\infty} \mu_\rho(E_k^{(\varepsilon)}) = \sum_{k=0}^{\infty} \langle 1_{B_\varepsilon^c}, \mathbb{P}_{\to k}\rho\rangle = \langle 1_{B_\varepsilon^c}, \bar{\rho}\rangle < \infty.$$

By Borel–Cantelli lemma, this implies $\mu_\rho(E^{(\varepsilon)}) = 0$, and therefore, $m(E^{(\varepsilon)}) = 0$ for all $\varepsilon > 0$. ■

Note that if the conditions in Lemma 1 are satisfied, then ρ is also integrable away from $\mathbf{0}$. This is because $\rho \leq \bar{\rho}$ due to the positivity of $\mathbb{P}_k$, $k \in \mathbb{Z}_{\geq 0}$.

The following lemma can be seen as the discrete-time counterpart of Theorem 1.

Lemma 2: Equation (15) is almost globally stable if there exist a positive constant $\alpha < 1$ and a sequence of positive functions $\rho_k \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}), k \in \mathbb{Z}_{\geq 0}$ dominated by a function $\rho_{\max} \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$, i.e., $\rho_k \leq \rho_{\max}, k \in \mathbb{Z}_{\geq 0}$, such that the following statements hold:

$\rho_{\max}$ is integrable away from $\mathbf{0}$, and

$\mathbb{P}_k\rho_k \leq \alpha\rho_{k+1}$ for all $k \in \mathbb{Z}_{\geq 0}$,

where $\mathbb{P}_k$ denotes the Frobenius–Perron operator of f_k restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ for each $k \in \mathbb{Z}_{\geq 0}$.

[2]Note that $\bar{\rho}$ is well defined as a function from $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to $\mathbb{R} \cup \{\infty\}$.

[3]In fact, μ_ρ and m are equivalent, i.e., they have the same measure zero set. This is because μ_ρ is absolute continuous with respect to m by its definition.

Proof: Define $\bar{\rho} := \sum_{k=0}^{\infty} \mathbb{P}_{\to k}\rho_0$. Note that $\mathbb{P}_k\rho_k < \alpha\rho_{k+1}$ implies $\mathbb{P}_{k+1}\mathbb{P}_k\rho_k \le \mathbb{P}_{k+1}\alpha\rho_{k+1} \le \alpha^2\rho_{k+2}$ due to the positivity of the Frobenius–Perron operator. Iterative application of this gives

$$\bar{\rho} \le \rho_0 + \alpha\rho_1 + \alpha^2\rho_2 + \cdots \le \ \rho_{\max}\frac{1}{1-\alpha}$$

which implies that $\bar{\rho}$ is integrable away from $\mathbf{0}$. Therefore, Lemma 1 for $\rho = \rho_0$ implies the result. ■

C. Discretizations of (2) for a Switching Signal

We assume that for any $\sigma \in \mathcal{S}_\tau$, almost all solutions of (2) exist for all $t > 0$. For a fixed $\sigma \in \mathcal{S}_\tau$, we consider the time ΔT maps $\Phi_i^{(\Delta T)}$, $i \in \mathbb{N}$ which map the states at time $i \cdot \Delta T$ to the states at time $(i+1)\cdot\Delta T$ under the dynamics of (2). This gives rise to a discrete-time nonautonomous system

$$x(k+1) = \Phi_k^{(\Delta T)}\left(x(k)\right), \quad k \in \mathbb{Z}_{\ge 0} \tag{16}$$

which produces solutions that are discretizations of the continuous-time trajectories of (2). Note that the system (16) for $\sigma \in \mathcal{S}_\tau$ is a discrete-time nonautonomous system in the form of (15), with the only difference being that the maps in (16) are defined almost everywhere. Nevertheless, Lemma 2 applies to (16) as the measurable functions and the Frobenius–Perron operators in this lemma are defined up to a set of measure zero.

Lemma 3: Assume that almost all solutions of (2) exist for $\sigma \in \mathcal{S}_\tau$. Then, the switched system (2) is almost globally stable for $\sigma \in \mathcal{S}_\tau$ if and only if its discretization (16) for σ is almost globally stable for all sufficiently small $\Delta T > 0$.

Proof: The necessity part of the proof is trivial. To show the sufficiency, let us choose a sequence of sufficiently small numbers $\{\Delta T_i\}_{i\in\mathbb{Z}_{>0}} \to 0$ such that the discretization (16) with sampling time ΔT_i is almost globally stable for all $i \in \mathbb{Z}_{>0}$. In other words, for each $i \in \mathbb{Z}_{>0}$, there exists a set N_i of zero Lebesgue measure such that all initial points in $\mathbb{R}^n \setminus N_i$ converge to $\mathbf{0}$ for the discretization (16) with sampling time ΔT_i. Set $N := \cup_i N_i$, which has zero Lebesgue measure. It is enough to show that a solution of (2), say $x(t)$, for a given $\sigma \in \mathcal{S}_\tau$ and an initial state $x(0) = x_0 \in \mathbb{R}^n \setminus N$ converges to $\mathbf{0}$ if its discretization $x(k\Delta T_i)$, namely the solution of (16) with sampling time ΔT_i for $x(0) = x_0$, converges to $\mathbf{0}$ for all $i \in \mathbb{Z}_{>0}$. We show this by contradiction as follows: Let us assume that $\lim_{k\to\infty} x(k\Delta T_i) = \mathbf{0}$ for all $i \in \mathbb{Z}_{>0}$ and $\lim_{t\to\infty} x(t) \ne \mathbf{0}$. The second assumption implies that there exists $\varepsilon > 0$ such that for each time T, there exists a larger time $T'(T)$ such that $x(T'(T)) \in B_\varepsilon^{\mathrm{c}}$, whereas the first assumption implies that there exists $k_1 \in \mathbb{Z}_{\ge 0}$ such that the sequence $x(k\Delta T_1)$ is contained in $B_{\varepsilon/2}^{\mathrm{c}}$ for all $k \ge k_1$. Hence, by the continuity of $x(t)$ with respect to t, we can choose an increasing sequence of time instants $\{t_k\}_{k\in\mathbb{Z}_{\ge 0}}$ such that $\|x(t_0)\| = \varepsilon/2$ where $t_0 > k_1\Delta T_1$, $\|x(t_k)\| = \varepsilon$ for all odd ks and $\|x(t_k)\| = \varepsilon/2$ for all even ks. This can be done as follows: Set $k = k_1$, consider $x(k\Delta T_1) \in B_{\varepsilon/2}$ and $x(T'(k\Delta T_1)) \in B_\varepsilon^{\mathrm{c}}$, and by continuity choose $t_0, t_1 \in (k\Delta T_1, T'(k\Delta T_1)]$ such that $\|x(t_0)\| = \varepsilon/2$ and $\|x(t_1)\| = \varepsilon$ (see Fig. 3). Repeat this process for $k = k_2$ satisfying $k_2\Delta T_1 > t_1$ to obtain t_2 and t_3 and so on. By the

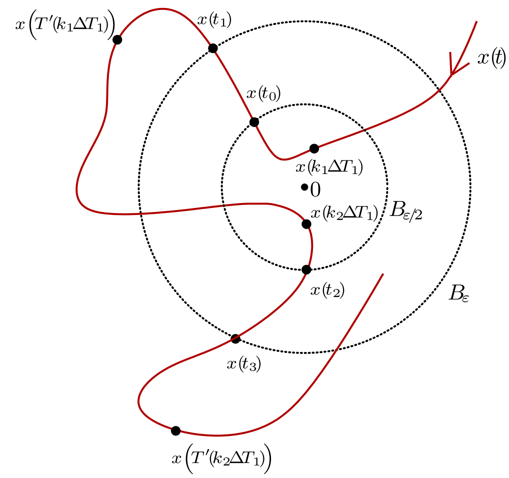

Fig. 3. Illustration for the proof of Lemma 3.

continuous differentiability of f_ps, there exists a common local Lipschitz constant L on B_ε valid for all f_ps. Considering $x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f_p(x(s))ds$ for an even k, and applying Gronwall inequality, we obtain $\|x(t_{k+1})\| \le \|x(t_k)\|\mathrm{e}^{\mathrm{L}(\mathrm{t_{k+1}}-\mathrm{t_k})}$. Then, we have $\liminf_k (t_{k+1} - t_k) \ge \frac{\ln(2)}{L} > 0$. Hence, one can choose a sufficiently small ΔT_j such that $x(k\Delta T_j)$ visits $B_\varepsilon \setminus B_{\varepsilon/2}$ infinitely often, which contradicts $\lim_{k\to\infty} x(k\Delta T_j) = \mathbf{0}$. ■

D. Monotonicity of the Lyapunov Density

We now state some technical lemmas that provide the required link between the properties of Lyapunov densities for continuous-time and discrete-time cases.

Lemma 4 (see [1]): Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^n$, $\rho : D \to \mathbb{R}$ be continuously differentiable functions with ρ being integrable. Let ϕ_t denote the solution map of $\dot{x} = f(x)$. For a measurable set Z, assume that $\phi_s(Z) = \{\phi_s(x) | x \in Z\}$ is a subset of D for all s between 0 and t. Then, we have

$$\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(x)dx = \int_0^t \int_{\phi_s(Z)} [\nabla\cdot(f\rho)](x)dxds.$$

Lemma 5: For a continuously differentiable vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(\mathbf{0}) = \mathbf{0}$, suppose that almost all solutions of $\dot{x} = f(x)$ exist for all $t > 0$. Assume that there exist a constant $\kappa > 0$ and a nonnegative, continuously differentiable function $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ such that the following statements hold.

1) $\rho(x)$ is integrable away from $\mathbf{0}$.
2) $\nabla\cdot(\rho f) \ge \kappa\rho$.

Then, for all $T > 0$

$$\mathbb{P}^{(T)}\rho \le \mathrm{e}^{-\kappa \mathrm{T}}\rho$$

where $\mathbb{P}^{(T)}$ is the Frobenius–Perron operator of the time T solution map of $\dot{x} = f(x)$.

Proof: Let $T > 0$ and $\mathcal{X}_T$ be the set of initial states for which the solution of $\dot{x} = f(x)$ exists for $t \in [0, T]$. $\mathcal{X}_T$ is an open subset of $\mathbb{R}^n$ [41, Th. 3.5] and by assumption, $\mathbb{R}^n \setminus \mathcal{X}_T$ is a zero measure set. Hence, it suffices to show that $\mathbb{P}^{(T)}\rho \le$

$e^{-\kappa T}\rho$ on $\mathcal{X}_T \setminus \{\mathbf{0}\}$. Let Z be an arbitrary compact subset in $\mathcal{X}_T \setminus \{\mathbf{0}\}$ and $\phi(t,x) = \phi_t(x)$ denote the flow map generated by the vector field f. The set $\phi([0,T],Z)$ is compact due to the continuity of $\phi(t,x)$ in both time and space variables. Thus, $\{\mathbf{0}\}$ and $\phi([0,T],Z)$ are disjoint closed subsets of $\mathbb{R}^n$. By the normality of $\mathbb{R}^n$, there exist a pair of disjoint open sets U and V such that $\{\mathbf{0}\} \subset U$, $\phi([0,T],Z) \subset V$. Therefore, ρ is integrable on V and Lemma 4 for $D = V$ implies

$$\int_{\phi_T(Z)} \rho(z)dz - \int_Z \rho(z)dz = \int_0^T \int_{\phi_s(Z)} [\nabla \cdot (f\rho)](z)dzds. \tag{17}$$

Using the Frobenius-Perron operator $\mathbb{P}^{(-s)}$ for the map $\phi_{-s} := (\phi_s)^{-1}$ for $s \in [0,T]$ and applying $\nabla \cdot (f\rho)(x) \geq \kappa\rho(x)$, we obtain

$$\int_Z \mathbb{P}^{(-T)}\rho(x)dx - \int_Z \rho(x)dx \geq \int_0^T \int_Z \kappa\mathbb{P}^{(-s)}\rho(x)dxds. \tag{18}$$

Since $\nabla \cdot (f\rho)(x) \geq \kappa\rho(x) > 0$, applying Lemma 4 again for $t = s$, where $0 \leq s \leq T$, we obtain

$$\int_Z \mathbb{P}^{(-s)}\rho(x)dx > \int_Z \rho(x)dx. \tag{19}$$

(18) and (19) imply

$$\int_Z \mathbb{P}^{(-T)}\rho(x)dx > \int_Z (1+\kappa T)\rho(x)dxds.$$

Finally, we have $\mathbb{P}^{(-T)}\rho(x) > (1+\kappa T)\rho(x)$, since Z is an arbitrary compact subset in $\mathcal{X} \setminus \{\mathbf{0}\}$. By using the positivity of the Frobenius-Perron operator, we obtain that $\mathbb{P}^{(T)}\rho(x) < \frac{1}{(1+\kappa T)}\rho(x)$. Dividing the interval $[0,T]$, into equal pieces, $\Delta t = \frac{T}{n}$, we obtain $\mathbb{P}^{(\Delta t)}\rho(x) < \frac{1}{(1+\frac{\kappa T}{n})}\rho(x)$. Then, for all $n \in \mathbb{Z}_{>0}$

$$\mathbb{P}^{(T)}\rho(x) = \left(\mathbb{P}^{(\Delta t)}\right)^n \rho(x) < \frac{1}{(1+\frac{\kappa T}{n})^n}\rho(x).$$

Taking the limit as $n \to \infty$, we get $\mathbb{P}^{(T)}\rho(x) \leq e^{-\kappa T}\rho(x)$. ■

E. Sufficient Condition via Frobenius–Perron Operators

The proof of Theorem 1 relies on a less-conservative lemma stated as follows.

Lemma 6: Consider the switched system (2). Assume that there exist constants $\tau_{\min} > 0$ and $\kappa_p > 0$, $p \in \{1,2,\ldots,N\}$ and nonnegative, continuously differentiable functions $\rho_p : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, $p \in \{1,2,\ldots,N\}$ such that the following conditions are satisfied for all $p, m \in \{1,2,\ldots,N\}$:

$$(1 + \|f_p(x)\|)\rho_p(x) \text{ is integrable away from } \mathbf{0} \tag{20}$$

$$\nabla \cdot (\rho_p f_p)(x) \geq \kappa_p \rho_p(x) \;\; \forall\, x \in \mathbb{R}^n \setminus \{\mathbf{0}\} \tag{21}$$

$$\mathbb{P}_p^{(\tau_{\min})}\rho_p(x) \leq \rho_m(x) \;\; \forall\, x \in \mathbb{R}^n \setminus \{\mathbf{0}\} \tag{22}$$

where $\mathbb{P}_p^{(t)}$ is the Frobenius–Perron operator of the time t map for the subsystem $\dot{x} = f_p(x)$. Then, the system (2) is almost globally stable for any $\tau > \tau_{\min}$.

Proof: Proposition 2 together with (20) and (21) implies that for each subsystem $\dot{x} = f_p(x)$, $p = 1,2,\ldots,N$, almost all solutions exist for all $t \geq 0$. Therefore, for the switched system (2) almost all solutions exist for all $t \geq 0$. To guarantee that almost all solutions converge to $\mathbf{0}$, in view of Lemma 3, we show that almost all solutions converge to $\mathbf{0}$ for the discretization (16) for all sufficiently small ΔT. This can be done by proving that the assumptions in Lemma 2 are satisfied for (16) for a sequence of density function labeled as

$$(\nu_k)_{k\in\mathbb{Z}_{>0}} = \left(\nu_1^{(1)},\ldots,\nu_1^{(K_1)},\nu_2^{(1)},\ldots,\nu_2^{(K_2)},\nu_3^{(1)},\ldots\right).$$

Here, K_i is the number of sampling instants in the interval $[t_{i-1}, t_i)$ (see Fig. 4).

Without loss of generality, we assume that the switching instants satisfy $t_i - t_{i-1} < 2\tau$ for all $i \in \mathbb{Z}_{\geq 0}$, since otherwise we could split the subsystem operation interval $[t_{i-1}, t_i)$ into pieces of length greater than or equal to τ by adding dummy switching instants in this interval that represent switchings from the subsystem f_{p_i} to the same subsystem f_{p_i}. This results in an upper bound on K_is as $K_i \leq K_{\max} := 2\tau/\Delta T$. Note that the first sampling instant in the interval $[t_{i-1}, t_i)$ is $\bar{K}_{i-1}\Delta T$, where $\bar{K}_i = K_1 + \cdots + K_i$ for all $i \in \mathbb{Z}_{>0}$ and $\bar{K}_0 = 0$ (see Fig. 4). We define $\nu_i^{(j)}$ recursively as follows:

1) $\nu_i^{(1)} = \mathbb{P}_{p_i}^{(\bar{K}_{i-1}\Delta T - t_{i-1})}\rho_{p_i}$,
2) $\nu_i^{(j+1)} = \mathbb{P}_{p_i}^{(\Delta T - \Delta\tau/K_i)}\nu_i^{(j)}$, for $j = 1,\ldots,K_i - 1$,

where $\Delta\tau := \tau - \tau_{\min}$. Note that recursive application of the above-mentioned yields

$$\nu_i^{(K_i)} = \mathbb{P}_{p_i}^{(\Delta t_i - \Delta\tau + \Delta\tau/K_i - (t_i - (\bar{K}_i - 1)\Delta T))}\rho_{p_i}. \tag{23}$$

Equation (20) implies that each ν_k is integrable away from $\mathbf{0}$. [42, Th. 1] implies $\rho_k(x) > 0$.[4] Hence, positivity of the Frobenius–Perron operator implies that $\nu_k > 0$. It remains to show that there exists $\alpha < 1$ such that for all k, $\mathbb{P}_k\nu_k \leq \alpha\nu_{k+1}$ almost everywhere, where $\mathbb{P}_k$ is the Frobenius–Perron operator of the map $\Phi_k^{(\Delta T)}$. We assume that ΔT is sufficiently small such that $\Phi_k^{(\Delta T)}$ either consists of a time ΔT map of a subsystem f_p or is a composition of two maps, a ΔT_1 map of a subsystem f_p and a ΔT_2 map of the next subsystem f_m, where $\Delta T_1 + \Delta T_2 = \Delta T$. In particular, for k not equal to any $\bar{K}_m$, $\Phi_k^{(\Delta T)}$ consists of one time ΔT map (see Fig. 4) and from Lemma 5, we have

$$\begin{aligned}\mathbb{P}_k\nu_k \;&=\; \mathbb{P}_{p_i}^{(\Delta T)}\nu_i^{(j)} \leq \mathbb{P}_{p_i}^{(\Delta\tau/K_i)}\nu_i^{(j+1)} \leq e^{-\kappa_{p_i}\Delta\tau/K_i}\nu_i^{(j+1)} \\ &\leq\; \alpha\nu_{k+1}\end{aligned}$$

where i is such that $[(k-1)\Delta T, k\Delta T) \subset [t_{i-1}, t_i)$, j is such that $k = \bar{K}_{i-1} + j$ and $\alpha := e^{-(\min_p \kappa_p)\Delta\tau/K_{\max}}$. For $k = \bar{K}_i$ for some i, $\Phi_k^{(\Delta T)}$ consists of two maps in general, as explained earlier, in particular with $\Delta T_1 = t_i - (\bar{K}_i - 1)\Delta T$ and $\Delta T_2 =$

[4]The requirement of the existence of all solutions in the proof of [42, Th. 1] can be replaced by the weaker requirement that almost all solutions exists for $t > 0$.

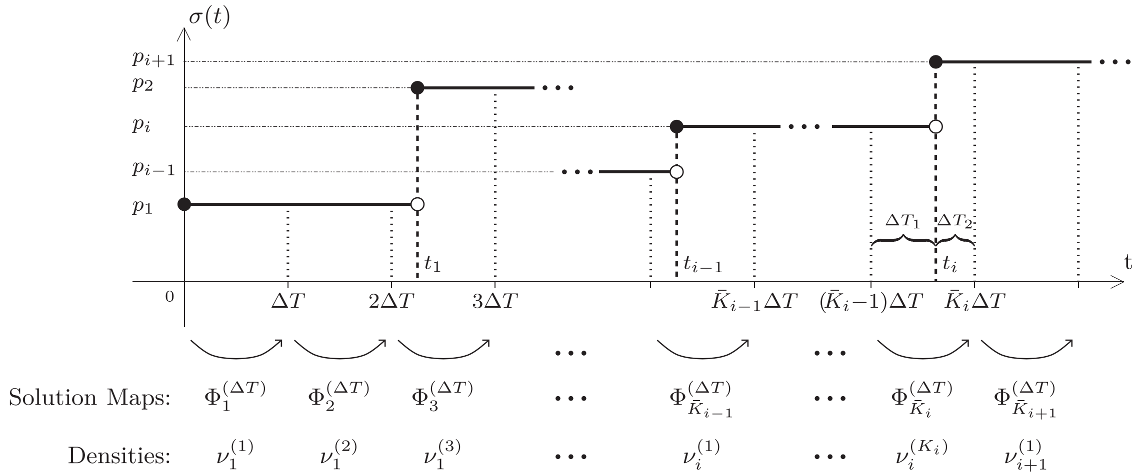

Fig. 4. Illustration for the proof of Lemma 6.

$\bar{K}_i \Delta T - t_i$. Using (23), we have

$$\begin{aligned}\mathbb{P}_k \nu_k &= \mathbb{P}_{p_{i+1}}^{(\Delta T_2)} \mathbb{P}_{p_i}^{(\Delta T_1)} \nu_i^{(K_i)} \\ &= \mathbb{P}_{p_{i+1}}^{(\bar{K}_i \Delta T - t_i)} \mathbb{P}_{p_i}^{(\Delta t_i - \Delta \tau)} \mathbb{P}_{p_i}^{(\Delta \tau / K_i)} \rho_{p_i}.\end{aligned}$$

Using $\Delta t_i - \Delta \tau \geq \tau_{\min}$ and invoking Lemma 5 for the last Frobenius–Perron operator above, we get

$$\begin{aligned}\mathbb{P}_k \nu_k &\leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i \Delta T - t_i)} \mathbb{P}_{p_i}^{(\tau_{\min})} e^{-\kappa_{p_i}(\Delta \tau / K_i)} \rho_{p_i} \\ &\leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i \Delta T - t_i)} \mathbb{P}_{p_i}^{(\tau_{\min})} \alpha \rho_{p_i}.\end{aligned}$$

Finally, (22) implies $\mathbb{P}_k \nu_k \leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i \Delta T - t_i)} \alpha \rho_{p_{i+1}} = \alpha \nu_{i+1}^{(1)} = \alpha \nu_{k+1}$. ■

F. Proof of Theorem 1

Existence of almost all solutions of the switched system follows as in the proof of Lemma 6. Assume that conditions (5)–(7) are satisfied for densities ρ_p^*, $p \in \{1, \ldots, N\}$. Equation (5) implies that ρ_p^*s are integrable away from $\mathbf{0}$. Let β_p^*, $p \in \{1, \ldots, N\}$ be the numbers for which the minimum in (8) is attained (the minimum is attained by [37, Th. 1.1]). Define $\tilde{\rho}_p := \beta_p^* \rho_p^*$ for $p \in \{1, \ldots, N\}$. Equation (7) implies $\frac{\beta_p^*}{\beta_m^*} c_{pm} \geq \frac{\tilde{\rho}_p}{\tilde{\rho}_m}$, and (8) implies $\tau_{\min} \geq \frac{\ln\left(\frac{\beta_p^*}{\beta_m^*} c_{pm}\right)}{\kappa_p} \geq \frac{\ln\left(\frac{\tilde{\rho}_p}{\tilde{\rho}_m}\right)}{\kappa_p}$, which leads to $e^{-\kappa_p \tau_{\min}} \tilde{\rho}_p \leq \tilde{\rho}_m$. Hence, by Lemma 5, $\mathbb{P}_p^{(\tau_{\min})} \tilde{\rho}_p \leq \mathrm{e}^{-\kappa \tau_{\min}} \tilde{\rho}_{\mathrm{p}} \leq \tilde{\rho}_{\mathrm{m}}$, and the proof follows by applying Lemma 6 for densities $\tilde{\rho}_p, p \in \{1, \ldots, N\}$.

IV. Some Remarks

We now remark on the monotonicity of multiple Lyapunov densities and on the generality of Lemma 6.

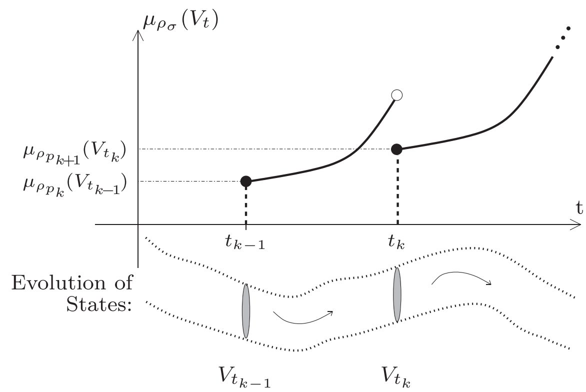

Fig. 5. Change of integrals of densities with time.

A. Monotonicity of Lyapunov Densities

Values of multiple Lyapunov functions decrease with time (along solutions) monotonically on each operating interval and from one switching instant to the next, allowing increases at switching instants. As an analogue to this, integrals of multiple Lyapunov densities increase with time (over a set of states) on each operating interval and from one switching instant to the next, allowing decreases at switching instants (see Fig. 5). To be precise, assume that the switched system operates initially with the subsystem f_{p_k} on the interval $[t_{k-1}, t_k)$ and then switches to the subsystem $f_{p_{k+1}}$ at the switching instant t_k. Recall that $\mu_\rho(V) := \int_V \rho dx$. Equation (21) and Lemma 5 imply that $\mathbb{P}_{p_k}^{(t)} \rho_{p_k} < \rho_{p_k}$ for $t > 0$. Integrating both sides over $\phi_t(V)$, where V is a measurable set of states and ϕ_t is the time t solution map of the subsystem f_p, we get $\int_{\phi_t(V)} \mathbb{P}_{p_k}^{(t)} \rho_{p_k} dx < \int_{\phi_t(V)} \rho_{p_k} dx$, which implies that $\int_V \rho_{p_k} dx < \int_{\phi_t(V)} \rho_{p_k} dx$. As a result, $\mu_{\rho_{p_k}}(V)$ increases on the interval $[t_{k-1}, t_k)$. On the other hand, since $t_k - t_{k-1} > \tau_{\min}$, we have

$$\begin{aligned}\mathbb{P}_{p_k}^{(t_k - t_{k-1})} \rho_{p_k} &= \mathbb{P}_{p_k}^{(\tau_{\min})} \mathbb{P}_{p_k}^{(t_k - t_{k-1} - \tau_{\min})} \rho_{p_k} \\ &< \mathbb{P}_{p_k}^{(\tau_{\min})} \rho_{p_k} \leq \rho_{p_{k+1}}\end{aligned}$$

where the first inequality follows from the positivity of the Frobenius–Perron operator and the second inequality follows from (22). Integrating both sides of the inequality $\mathbb{P}_{p_k}^{(t_k-t_{k-1})}\rho_{p_k} < \rho_{p_{k+1}}$ over $V_{t_k} := \phi_{t_k-t_{k-1}}(V_{t_{k-1}})$ for a measurable set $V_{t_{k-1}}$, we have $\int_{V_{t_k}} \mathbb{P}_{p_k}^{(t_k-t_{k-1})}\rho_{p_k}\,dx < \int_{V_{t_k}} \rho_{p_{k+1}}\,dx$, which implies $\int_{V_{t_{k-1}}} \rho_{p_k}\,dx < \int_{V_{t_k}} \rho_{p_{k+1}}\,dx$. Therefore, we obtain $\mu_{\rho_{p_k}}(V_{t_{k-1}}) < \mu_{\rho_{p_{k+1}}}(V_{t_k})$ meaning that integrals of densities $(\mu_{\rho_{\sigma(t)}})$ increase with time from one switching instant to the next, which is depicted in Fig. 5.

B. Special Case: Linear Switched Systems

For linear switched systems with stable subsystems, Lemma 6 generalizes an LMI condition based on multiple quadratic Lyapunov functions [35]. Consider

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma \in \mathcal{S}_\tau, \quad t \in [0,\infty) \tag{24}$$

where A_ps are Hurwitz matrices. The sufficient condition obtained in [35] for the exponential stability of (24) is that there exist $\tau_{\min} < \tau$ and positive definite, symmetric matrices $P_1,\ldots,P_N$ such that

$$A_p^{\mathrm{T}}P_p + P_pA_p < 0, \quad p \in \{1,\ldots,N\} \tag{25}$$

$$e^{A_p^{\mathrm{T}}\tau_{\min}}P_m e^{A_p\tau_{\min}} < P_p, \quad p,m \in \{1,\ldots,N\}. \tag{26}$$

Let us consider densities for each subsystem as

$$\rho_p = (x^{\mathrm{T}}P_px)^{-\gamma} \tag{27}$$

where P_ps are $n\times n$ positive definite symmetric matrices and $\gamma \geq 2$ is sufficiently large. We want to show that for this particular choice of densities, Lemma 6 generalizes conditions (25) and (26). The integrability condition (20) in Lemma 6 is satisfied since $\|f_p(x)\|\rho_p(x) = \|A_px\|(x^{\mathrm{T}}P_px)^{-\gamma}$ is of the same order as $\|x\|^{-(2\gamma-1)} \leq \|x\|^{-3}$. In order to see that (25) implies (21), observe that (25) implies

$$\exists\epsilon > 0 : x^{\mathrm{T}}\left(A_p^{\mathrm{T}}P_p + P_pA_p\right)x \leq -\epsilon x^{\mathrm{T}}P_px. \tag{28}$$

Set $\gamma := \frac{\kappa_p - \mathrm{trace}(A_p)}{\epsilon}$, which is positive, because $\mathrm{trace}(A_p) < 0$ by the stability of subsystems. Multiplying both sides of (28) by $-\gamma\left(x^{\mathrm{T}}P_px\right)^{-\gamma-1}$, we get

$$-\gamma(x^{\mathrm{T}}P_px)^{-\gamma-1}x^{\mathrm{T}}\left(A_p^{\mathrm{T}}P_p+P_pA_p\right)x+(x^{\mathrm{T}}P_px)^{-\gamma}\mathrm{trace}(A_p) \geq \kappa_p(x^{\mathrm{T}}P_px)^{-\gamma}.$$

which is equivalent to (21) under (27), in view of $\nabla\cdot(\rho_pf_p) = \nabla\rho_pf_p + \rho_p\nabla\cdot f_p$, where $f_p = A_px$ and $\nabla\cdot f_p = \mathrm{trace}(A_p)$. Note that γ can be made arbitrary large as ϵ in (28) can be chosen arbitrarily small, therefore the integrability condition is not violated. Now, assume that (26) is satisfied for some $\tau_{\min} < \tau$. Then, there exists $\beta \in (0,1)$ arbitrarily close to 1 such that

$$x^{\mathrm{T}}e^{A_p^{\mathrm{T}}\tau_{\min}}P_m e^{A_p\tau_{\min}}x \leq \beta x^{\mathrm{T}}P_px. \tag{29}$$

Set $\gamma := \frac{-\mathrm{trace}(A_p)\tau_{\min}}{\ln\beta}$. Then, (29) implies

$$x^{\mathrm{T}}e^{A_p^{\mathrm{T}}\tau_{\min}}P_m e^{A_p\tau_{\min}}x \leq e^{\frac{\mathrm{trace}(A_p)\tau_{\min}}{\gamma}}x^{\mathrm{T}}P_px. \tag{30}$$

Applying the state transformation $x = e^{-A_p\tau_{\min}}\hat{x}$ and taking to the power $-\gamma$, we get

$$\left(\hat{x}^{\mathrm{T}}e^{-A_p^{\mathrm{T}}\tau_{\min}}P_pe^{-A_p\tau_{\min}}\hat{x}\right)^{-\gamma}\det(e^{-A_p\tau_{\min}}) \leq \left(\hat{x}^{\mathrm{T}}P_m\hat{x}\right)^{-\gamma}$$

which is equivalent to (22) under (27), in view of (14). Note that γ can be made arbitrary large by choosing β sufficiently close to one, therefore γs in the aforementioned discussion can be chosen identically.

V. Conclusion

We have derived sufficient conditions for the almost global stability of nonlinear switched systems with time-dependent switching. Our method is based on multiple Lyapunov densities and can be seen as the analogue of the multiple Lyapunov function technique, for the framework of almost global stability.

After this paper, new directions in the field of almost global stability may open up. First, the use of Lyapunov densities for the verification of temporal properties, such as safety, reachability, eventuality, and avoidance, studied in [20], can be considered for switched nonlinear systems. Second, following the ideas presented in [43] and [44] for the graph-based estimations of the average dwell time, the techniques in this paper can be used to obtain graph-based estimations of the average dwell time for the almost global stability of nonlinear switched systems.

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